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ON THE SINGULARITY INDUCED BY CERTAIN MIXED BOUNDARY CONDITIONS IN LINEARIZED AND NONLINEAR ELASTOSTATICS†

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Abstract—This study concerns the local character of the elastostatic field in plane strain near a point that separates a free from an adjoining fixed segment of a rectilinear boundary-component. The well-known singular field behavior predicted by the linear theory, as such a point is approached, exhibits oscillatory deformations and stresses. It is shown here by means of an asymptotic analysis that the foregoing anomalous behavior does not occur within the nonlinear theory of harmonic elastic materials. In preparation for this task certain general aspects of the latter theory are reviewed. The results obtained in the nonlinear asymptotic treatment of the class of mixed boundary-value problems considered are discussed in detail with particular attention to the problem of a bonded flat-ended rigid punch.

INTRODUCTION

This paper is a sequel to two earlier studies [1, 2] concerning the implications of nonlinear elastostatics in singular boundary-value problems for which the linear theory—in conflict with its underlying approximative assumptions—predicts locally unbounded infinitesimal strains and stresses. The earlier investigations referred to pertain to the deformations and stresses near the tip of a crack under conditions of plane strain. The present work aims at the plane-strain field behavior in the vicinity of a point that separates a load-free from an adjoining and collinear fixed portion of the boundary of an elastic body.

A familiar example of a mixed boundary-value problem of this type is furnished by the particular problem of a rigid flat-ended punch that is bonded to the straight edge of a semi-infinite elastic slab and subjected to an axial load.‡ The known solution of this problem within the linearized theory of plane strain exhibits strikingly anomalous singularities at the punch corners: the surface displacements along the two free boundary-components and both contact stresses are oscillatory in a neighborhood of the corners, the stresses becoming unbounded as either corner is approached. As a consequence of this state of affairs the problem of the "rough punch", in which the given load is compressive and the complete bond is assumed to be supplied solely by the available friction, has no solution in its conventional formulation on the basis of the linear theory of elasticity.

It is clear from an investigation due to Williams [3], which systematically generalizes a scheme apparently originated by Knein [4], that the peculiar oscillations predicted by the solution of the bonded-punch problem are in fact characteristic of all linear mixed boundary-value problems belonging to the category under consideration.

The foregoing observations suggest the question as to whether or not the pathological features recalled above are due strictly to the linearization of the problem or whether they arise also in the finite theory of elasticity, which admits large deformations and takes account of constitutive nonlinearities as well. In attempting to cope with this issue it is neither necessary nor feasible to insist on broad generality as far as the governing constitutive law is concerned. We therefore confine our attention to the analytically amenable class of harmonic elastic materials and introduce certain additional restrictions regarding the response of the material to homogeneous plane deformations. We then show by asymptotic means that—at least within the limitations of the present study—the nonlinear theory fails to predict the oscillatory singular behavior arising in the linearized theory.§

In Section 1 we outline the local two-dimensional analysis within the linear theory of the elastostatic field near the transition-point of a fixed-free rectilinear boundary. We then discuss

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‡As explained in Section 1, the translation of the contact-segment is removable without loss of generality.

§Actually our results remain valid for an elastic material that is merely "asymptotically harmonic" in a sense described at the end of Section 3.

the results thus obtained, which are included among the "corner singularities" deduced in [3], and establish their consistency with the pertinent asymptotic behavior of the known global solution to the specific problem of the bonded punch.

In Section 2 we first recall some relevant results from the nonlinear equilibrium theory of plane strain for compressible, homogeneous and isotropic, hyperelastic solids. The remainder of this section is devoted to the special class of elastic materials of harmonic type. Here we discuss the response of such solids to a plane pure homogeneous deformation. Finally, at the end of Section 2, we cast the field equations and constitutive relations governing plane deformations of harmonic materials into an economical complex form.

Section 3 is concerned with the analogue in the finite theory of harmonic materials of the class of mixed boundary-value problems dealt with in Section 1 on the basis of the linearized theory. The analysis carried out in Section 3 leads to an asymptotic solution up to third order, which is contingent upon certain hypotheses as to the local structure of the unknown global solution.

The results for the dominant behavior of the deformations and stresses emerging from the preceding asymptotic treatment are assembled and discussed in detail in Section 4. These results contain the real and imaginary parts of a complex amplitude parameter, the determination of which eludes the local analysis of the problem. At the end of Section 4 we cite an example of some practical interest in which the modulus of the amplitude parameter may be found directly from the data of the problem by means of an available conservation law. Guided by this example we then deduce a small-load estimate for the amplitude modulus appropriate to the nonlinear punch problem.

1. SINGULAR SOLUTIONS IN LINEARIZED PLANE STRAIN FOR THE HALF-PLANE WITH A FIXED-FREE BOUNDARY

With a view toward studying the singularities arising at angular corners in the two-dimensional linear equilibrium theory of homogeneous and isotropic elastic solids, Williams [3] deduced a sequence of elastostatic fields (appropriate to plane strain or generalized plane stress) for a wedge-shaped domain on the assumption that both legs of the boundary are free of tractions, both are held fixed, or one leg is free while the other is fixed. For our purposes it is sufficient to cite here a formulation of the problem considered in [3] and the results obtained there for the special case of the half-plane and mixed boundary conditions.

To this end let (x_1, x_2) be rectangular cartesian coordinates, introduce polar coordinates (r, θ) by setting

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \tag{1.1}$$

take $\mathscr{R}$ to be the region defined by

$$\mathscr{R} = \{(r, \theta) | 0 < r < \infty, \quad 0 \le \theta \le \pi\}, \tag{1.2}$$

and call $\mathring{\mathscr{R}}$ the interior of $\mathscr{R}$. Next, let u_α and $\sigma_{\alpha\beta}$ denote the cartesian components of the displacement vector-field and the stress tensor-field,† both of which are required to be suitably smooth on $\mathscr{R}$. The equilibrium and compatibility conditions then reduce to

$$\sigma_{\alpha\beta} = \epsilon_{\rho\alpha}\epsilon_{\tau\beta}\chi_{,\rho\tau}, \quad \nabla^4\chi = 0 \quad \text{on} \quad \mathring{\mathscr{R}}, \tag{1.3}‡$$

where $\epsilon_{\rho\alpha}$ are the components of the two-dimensional alternator and χ is the generating (biharmonic) Airy stress-function. Further, for *plane strain*, the stress-displacement relations become

$$\sigma_{\alpha\beta} = 2\mu\left[\frac{\nu}{1-2\nu}\,\delta_{\alpha\beta}u_{\gamma,\gamma} + u_{(\alpha,\beta)}\right] \quad \text{on} \quad \mathring{\mathscr{R}}, \tag{1.4}§$$

in which $\delta_{\alpha\beta}$ is the Kronecker delta, whereas μ and ν stand for the shear modulus and Poisson's

†Greek subscripts range over the integers (1, 2) and summation over repeated subscripts is taken for granted; subscripts preceded by a comma indicate partial differentiation with respect to the corresponding cartesian coordinates.

‡Throughout this paper we assume the absence of body forces.

§Here $u_{(\alpha,\beta)} = (1/2)(u_{\alpha,\beta} + u_{\beta,\alpha})$.

ratio. For *generalized plane stress*, u_α and $\sigma_{\alpha\beta}$ represent the appropriate thickness averages of displacement and stress; in this instance ν in (1.4) is to be replaced by $\nu/(1+\nu)$. Thus we need to discuss merely plane strain. In either case the boundary conditions at present take the form

$$\sigma_{\alpha 2}(r,0)=0, \quad u_\alpha(r,\pi)=0 \quad (0<r<\infty). \tag{1.5}$$

Entering the second of (1.3) with

$$\chi(r,\theta)=r^{m+1}F(\theta), \tag{1.6}$$

where m is a constant and F an unknown function, one finds that χ is an arbitrary linear combination of the four biharmonic functions

$$r^{m+1}\cos(m\pm 1)\theta, \quad r^{m+1}\sin(m\pm 1)\theta. \tag{1.7}$$

Upon subjecting the stresses generated by χ through the first of (1.3) and their associated displacements, which follow from the integration of (1.4), to the homogeneous boundary conditions (1.5), one is ultimately led to:

$$\sin^2 m\pi=\frac{4(1-\nu)^2}{3-4\nu}, \tag{1.8}$$

$$F(\theta)=a\left\{\frac{1-2\nu}{2(1-\nu)}\tan m\pi\left[\sin(m-1)\theta-\frac{m-1}{m+1}\sin(m+1)\theta\right]\right.$$
$$\left.+\cos(m-1)\theta-\cos(m+1)\theta\right\}, \tag{1.9}$$

where a is an arbitrary constant.

Since $-1<\nu<1/2$, the characteristic eqn (1.8) fails to possess real roots. Indeed, the roots of (1.8) may be arranged in a doubly infinite sequence of complex conjugate pairs in accordance with

$$m=j+\frac{1}{2}\pm i\eta \;(j=0,\pm1,\pm2,\ldots), \quad \eta=\frac{1}{2\pi}\log\kappa, \quad \kappa=3-4\nu. \tag{1.10}$$

On admitting complex values of the amplitude parameter a and thus setting

$$a=a_1+ia_2 \quad (a_1, a_2 \text{ real}), \tag{1.11}$$

one sees that each fixed value of j in (1.10) gives rise to a two-parameter family of elastostatic fields with the requisite properties, each such family being generated by the real-valued Airy function

$$\chi(r,\theta)=r^{j+3/2}\,Re\{\exp(i\eta\log r)\,F(\theta)\}, \tag{1.12}$$

which—through F—depends on the two arbitrary real amplitude parameters a_1 and a_2. Here and in what follows F is understood to be given by (1.9), while m has been chosen as

$$m=j+\frac{1}{2}+i\eta \quad (j=0,\pm1,\pm2,\ldots). \tag{1.13†}$$

The displacements generated by χ may be written as

$$2\mu u_\alpha(r,\theta)=r^{j+1/2}Re\{\exp(i\eta\log r)\,U_\alpha(\theta)\}, \tag{1.14}$$

†Clearly, $m=j+(1/2)-i\eta$ leads to solutions linearly dependent upon those corresponding to (1.13).

provided the complex-valued functions U_α are defined by means of

$$U_\alpha = [-(m+1)F + (1-\nu)G']c_\alpha + [F' - (1-\nu)(m-1)G]\epsilon_{\alpha\beta}c_\beta, \tag{1.15}$$

where the primes denote differentiation,

$$c_1 = \cos\theta, \quad c_2 = \sin\theta, \tag{1.16}$$

and G is given by

$$G(\theta) = \frac{4a}{m-1}\left\{\sin(m-1)\theta - \frac{1-2\nu}{2(1-\nu)}\tan m\pi \cos(m-1)\theta\right\}. \tag{1.17}$$

The associated field of stress, in turn, admits the representation

$$\sigma_{\alpha\beta}(r,\theta) = r^{j-1/2} Re\{\exp(i\eta \log r) S_{\alpha\beta}(\theta)\}, \tag{1.18}$$

if

$$S_{\alpha\beta} = F'' c_\alpha c_\beta - m(\epsilon_{\rho\alpha}c_\beta + \epsilon_{\rho\beta}c_\alpha)c_\rho F' + [m(m+1)\delta_{\alpha\beta} - (m^2-1)c_\alpha c_\beta]F. \tag{1.19}$$

To save space we refrain from recording the fully explicit representations (in terms of real-valued elementary functions exclusively) for u_α and $\sigma_{\alpha\beta}$, which—though quite lengthy—are readily deduced from (1.14) and (1.18) with the aid of (1.9), (1.13), (1.15), (1.16), (1.17), as well as (1.19). It is clear that each component of displacement and stress is a linear combination of terms containing a factor

$$\cos(\eta \log r) \quad \text{or} \quad \sin(\eta \log r). \tag{1.20}$$

The occurrence of these oscillatory functions of the radial coordinate is traceable to the absence of real roots of the characteristic eqn (1.8). Evidently, the preceding elastostatic fields become progressively more singular at the origin with decreasing values of the integer j. Moreover, according to (1.14) and (1.18), the displacement field remains bounded as $r \to 0$ if and only if $j \geq 0$, whereas the same is true of the stress field if and only if $j \geq 1$. It follows that the only member of the sequence of elastostatic fields under discussion that possesses bounded displacements and unbounded stresses at the origin is the one corresponding to $j = 0$.

The results cited above may be immediately generalized to the case in which the homogeneous displacement boundary condition at $\theta = \pi$ in (1.5) is replaced by the assignment of an arbitrary infinitesimal rigid displacement to this leg of the boundary. In that event the displacement field (1.14) needs to be augmented by addition of the appropriate rigid displacement field, whereas the stresses (1.18) remain unaltered.

It should be emphasized that (1.3), (1.4), together with (1.5), do not constitute a complete statement of a plane-strain problem for the half-plane. In particular no restrictions have been placed on the nature of the displacements or stresses at infinity or on the order of their singularities admitted at the origin. Although the resulting fields are *global* solutions of the equations governing the linear equilibrium theory of plane strain, their chief significance stems from the expectation that they characterize the possible *local* elastostatic field behavior in the vicinity of a point that separates two adjoining portions of a rectilinear boundary-segment, one of which is traction-free while the other is held fixed.

We turn next to a specific example illustrating the relevance of the foregoing results. Thus we consider the plane-strain problem for the half-plane arising if a finite segment of the boundary is forced to undergo a translation at right angles to itself by means of a flat-ended rigid "punch" that is fully bonded to this boundary-segment and subjected to an axial load (see Fig. 1); the remainder of the boundary, as well as infinity, are to be free of tractions. In preparation for a mathematical formulation of the problem just described we let $\mathcal{R}$ at present denote the half-plane $-\infty < x_1 < \infty$, $0 \leq x_2 < \infty$ with the exception of the two boundary points $(-l, 0)$ and $(l, 0)$ (punch corners). One is to find fields u_α and $\sigma_{\alpha\beta}$, symmetric about the x_2-axis and suitably smooth on $\mathcal{R}$,

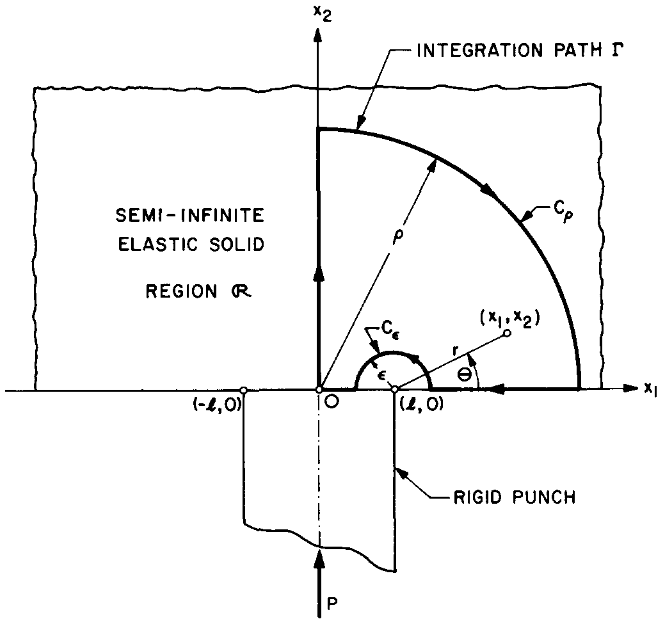

Fig. 1. Punch problem, geometry and coordinates.

such that (1.3), (1.4) hold subject to the following boundary and regularity conditions:

$$\sigma_{\alpha 2}(x_1, 0) = 0 \quad (l < |x_1| < \infty), \quad u_\alpha(x_1, 0) = 0 \quad (-l < x_1 < l), \tag{1.21†}$$

$$\sigma_{\alpha\beta}(x_1, x_2) = o(1) \quad \text{as} \quad r \to \infty, \tag{1.22}$$

$$u_\alpha(x_1, x_2) = O(1), \quad \sigma_{\alpha\beta}(x_1, x_2) = O(r^{-\delta}) \quad \text{as} \quad r \to 0 \quad (\delta < 1), \tag{1.23}$$

where δ is a constant and r now is the distance from $(l, 0)$. In addition one has the requirement

$$\int_{-l}^{l} \sigma_{22}(x_1, 0)\,dx_1 = -P, \tag{1.24}$$

provided P is the given scalar punch load—taken positive for an indenting punch. Note that the convergence of the integral in (1.24) is assured by (1.23). The solution to the above problem is unique because the homogeneous problem corresponding to $P = 0$ admits only the null solution, as may be confirmed by an elementary extension of the usual energy argument.‡

If the displacement boundary condition in (1.21) is generalized by setting

$$u_1(x_1, 0) = 0, \quad u_2(x_1, 0) = k \quad (-l < x_1 < l), \tag{1.25}$$

in which k is an arbitrarily prescribed constant, the solution to the problem so modified is trivially obtainable from the solution of the original problem by superposition of a rigid-body translation upon the displacement field. Such would no longer be the case if (1.22) too could be amended by demanding that the displacements—rather than the stresses—vanish at infinity. Unfortunately, however, the problem thus arrived at has no solution since the displacements are known beforehand to become logarithmically unbounded as $r \to \infty$, unless $P = 0$. Consequently, the "indentation" produced by a given punch load is inherently indeterminate and thus the normalization of u_2 underlying the second of (1.21) entails no loss in generality. It should be noted that the indeterminacy here alluded to is characteristic of two-dimensional punch and contact problems for a semi-infinite elastic solid, but is absent from the analogous three-dimensional problems.

†For the time being the displacements and stresses are regarded as functions of the cartesian coordinates.

‡In this connection one needs to invoke (1.23) and recall that for self-equilibrated contact tractions and vanishing stresses at infinity, $u_\alpha(x_1, x_2) = O(r^{-1})$, $\sigma_{\alpha\beta}(x_1, x_2) = O(r^{-2})$ as $r \to \infty$.

When $P > 0$ and the punch is assumed to adhere to the surface of the elastic solid solely by virtue of the available friction ("rough-indenter problem"), the requirements (1.21), (1.22), (1.23), (1.24) must be supplemented by the condition of unilateral constraint,

$$\sigma_{22}(x_1, 0) \leq 0 \quad (-l < x_1 < l). \tag{1.26}$$

The problem of the bonded punch stated earlier was apparently solved first by Abramov[5] with the aid of the Mellin transform. Later on Muskhelishvili[6]† recovered Abramov's solution by specializing a general scheme for the reduction to a Hilbert problem of the mixed half-plane problem in linear elastostatics. We cite here from[5], [7] merely the results for the contact stresses, which are found to be given by

$$\left.\begin{aligned}\sigma_{22}(x_1, 0) &= \frac{-P}{2\pi\sqrt{(l^2 - x_1^2)}}\frac{1+\kappa}{\sqrt{\kappa}}\cos\left[\eta\log\frac{l+x_1}{l-x_1}\right] \quad (-l < x_1 < l),\\ \sigma_{12}(x_1, 0) &= \frac{-P}{2\pi\sqrt{(l^2 - x_1^2)}}\frac{1+\kappa}{\sqrt{\kappa}}\sin\left[\eta\log\frac{l+x_1}{l-x_1}\right] \quad (-l < x_1 < l),\end{aligned}\right\} \tag{1.27‡}$$

where κ and η are the auxiliary functions of Poisson's ratio introduced in (1.10). On referring the stresses to the polar coordinates displayed in Fig. 1 and defined by

$$x_1 - l = r\cos\theta, \quad x_2 = r\sin\theta \quad (0 < r < \infty, 0 \leq \theta \leq \pi), \tag{1.28}$$

one draws from (1.27) that

$$\left.\begin{aligned}\sigma_{22}(r, \pi) &\sim -\frac{P(1+\kappa)}{2\pi\sqrt{(2\kappa l)}}r^{-1/2}\cos[\eta\log(r/2l)] \quad \text{as} \quad r \to 0,\\ \sigma_{12}(r, \pi) &\sim \frac{P(1+\kappa)}{2\pi\sqrt{(2\kappa l)}}r^{-1/2}\sin[\eta\log(r/2l)] \quad \text{as} \quad r \to 0.\end{aligned}\right\} \tag{1.29}$$

Here and throughout the remainder of this paper the asymptotic-equality symbol "$\sim$" is used in its standard connotation: thus (1.29) assert that their right and left-hand members differ from each other by functions of order $o(r^{-1/2})$.

In view of the mixed boundary conditions (1.21) and since the solution to the bonded-punch problem involves bounded displacements but unbounded stresses at the corners of the punch, one would anticipate its dominant asymptotic behavior, as $r \to 0$, to be furnished by (1.14) and (1.18) and their supporting equations, with $j = 0$ and for an appropriate choice of the complex amplitude parameter a entering (1.9) and (1.17). This expectation is borne out by a direct computation, which yields

$$a = -\frac{iP}{2\pi\sqrt{(2l)}}\left[\frac{1}{4} + \eta^2\right]^{-1/2}\exp[-i(\gamma + \eta\log 2l)], \quad \gamma = \tan^{-1}(2\eta). \tag{1.30}$$

In particular, this choice of the constant a leads to stresses $\sigma_{22}(r, \theta)$ and $\sigma_{12}(r, \theta)$ that coincide with the respective right member in (1.29) when $\theta = \pi$.

According to (1.10), $\kappa \to 1$ and $\eta \to 0$ as $\nu \to 1/2$, whence (1.27) implies

$$\sigma_{22}(x_1, 0) \to -\frac{P}{\pi\sqrt{(l^2 - x_1^2)}}, \quad \sigma_{12}(x_1, 0) \to 0 \quad \text{as} \quad \nu \to \frac{1}{2} \quad (-l < x_1 < l). \tag{1.31}$$

The normal contact tractions emerging in this limit, which conform to (1.26) when P is positive, are found to be identical with those deduced by Sadowsky[8] for the analogous problem of the ideally smooth indenter. It therefore follows from (1.31) that the solutions appropriate to the

†See also [7], p. 466.
‡The factor 1/2 is missing in the corresponding results on p. 466 of[7], which contain a misprint.

bonded and the smooth indenter coalesce in the limiting case of the *incompressible* elastic solid, as observed by Abramov[5].†

On the other hand, for a *compressible* (linearly elastic) material one has $\nu < 1/2$ and hence $\kappa > 1$, $\eta > 0$ on account of (1.10). In these circumstances (1.27) reveal that both contact tractions not only become unbounded as $x_1 \to \pm l$ but *undergo infinitely many sign reversals in a neighborhood of either punch-corner.*‡ The zeros of the normal stress $\sigma_{22}(x_1, 0)$ that are closest to the axis of symmetry evidently occur at

$$x_1 = \pm l_*, \quad l_* = l \tanh(\pi^2/2 \log \kappa) \tag{1.32}$$

and

$$\operatorname{sgn} \sigma_{22}(x_1, 0) = -\operatorname{sgn} P \quad (-l_* < x_1 < l_*). \tag{1.33}$$

The oscillatory behavior of the contact stresses described above reflects a property of the entire solution to the problem under consideration: all non-vanishing stresses and displacements are found to exhibit an infinity of sign changes near the corners of the punch. Moreover, this conclusion is consistent with the oscillatory behavior (as $r \to 0$) of the local approximating solution, pointed out in connection with (1.20).

We return now to the discussion of the normal contact traction given by the first of (1.27). If Poisson's ratio obeys $0 \le \nu < 1/2$, as is realistic for an actual compressible material, then the smallest value of l_* furnished by (1.32) corresponds to $\nu = 0$, in which case $\kappa = 3$; further,

$$\min l_* \doteq 0{\cdot}9997 l. \tag{1.34}$$

The oscillations of $\sigma_{22}(x_1, 0)$ are thus seen to be confined to exceedingly narrow boundary layers adjacent to the endpoints of the contact zone. The occurrence of any such sign reversals, however localized, nevertheless has some rather startling consequences: it leads to the prediction that a *tensile* load ($P < 0$) applied to the bonded punch produces contact tractions that are partly *compressive* at points sufficiently close to the punch corners, whereas a *compressive* load ($P > 0$) gives rise to tractions that are in part *tensile*. By the same token it follows from these observations that the solution of the bonded-punch problem when $P > 0$ violates the condition of unilateral constraint (1.26). Consequently the problem of the rough indenter has no solution in its conventional formulation, cited earlier. Put in physical terms, this conclusion implies that a rigid flat-ended indenter, pressed against a semi-infinite elastic solid in the presence of sufficient friction to prevent any lateral surface displacement within the contact zone, could not possibly maintain full contact with the indented solid.

Since the solution to the problem of the bonded punch involves unbounded stresses, and hence also unbounded strains, at the corners of the punch, it is in conflict with one of the approximative assumptions underlying its derivation on the basis of the linearized theory of elasticity. It is therefore natural to wonder how the anomalous behavior near the punch-corners predicted by the linear theory is modified when finite deformations are taken into account. This question supplies the motivation of the present study, in which we explore some implications of the fully nonlinear theory of compressible elastic media as to the nature of the singularities induced by mixed boundary conditions of the type arising in the punch problem.

2. FINITE PLANE ELASTOSTATIC STRAIN FOR COMPRESSIBLE HARMONIC MATERIALS. COMPLEX FORMULATION.

In this section we first summarize some prerequisites from the nonlinear equilibrium theory of plane strain for a compressible ideally elastic solid and subsequently consider the special case of harmonic materials.

Let $\mathscr{D}$ be the domain of the (x_1, x_2)-plane occupied by the open middle cross-section of a cylindrical or prismatic body in its undeformed configuration. Following the notation adopted

†This conclusion is also immediate from the classical solution for the problem of the half-plane under a concentrated normal load, which predicts vanishing tangential displacements at the boundary when $\nu = 1/2$.

‡A strictly analogous singular behavior prevails in the axisymmetric counterpart of the punch problem under discussion. See Keer[9] for references to the literature on this three-dimensional problem.

in [2],† we assume a plane deformation of the body, parallel to the (x_1, x_2)-plane, to be given by a transformation

$$y_\alpha = \hat{y}_\alpha(x_1, x_2) = x_\alpha + u_\alpha(x_1, x_2) \quad \text{for all} \quad (x_1, x_2) \in \mathscr{D}, \tag{2.1}$$

which maps $\mathscr{D}$ onto a domain $\mathscr{D}^*$ of the same plane. Accordingly, x_α and y_α are the material and the spatial coordinates, respectively, while u_α are the components in the underlying rectangular reference frame of the displacement vector $\mathbf{u}$. We require the mapping (2.1) to be at least twice continuously differentiable on $\mathscr{D}$ and to possess an inverse of the same degree of smoothness on $\mathscr{D}^*$.

Let $\mathbf{F}$ be the deformation-gradient tensor associated with the deformation (2.1) and J its Jacobian determinant, whence

$$F_{\alpha\beta} = \frac{\partial \hat{y}_\alpha}{\partial x_\beta} = \frac{\partial u_\alpha}{\partial x_\beta} + \delta_{\alpha\beta}, \quad J = \det \mathbf{F} > 0 \text{ on } \mathscr{D}; \tag{2.2}$$

further, let $\mathbf{G}$ be the (symmetric, positive-definite) deformation tensor defined through

$$\mathbf{G} = \mathbf{F}^T\mathbf{F} \quad \text{on} \quad \mathscr{D}. \tag{2.3‡}$$

Then the two fundamental scalar invariants of $\mathbf{G}$ may be taken as

$$I = tr\mathbf{G} = F_{\alpha\beta}F_{\alpha\beta}, \quad J = \sqrt{(\det \mathbf{G})} = F_{11}F_{22} - F_{12}F_{21}. \tag{2.4}$$

Next, let $\boldsymbol{\tau}$ stand for the two-dimensional Cauchy stress-tensor field, regarded as a function of position on $\mathscr{D}^*$, and call W the strain-energy density per unit undeformed volume as a function of position on $\mathscr{D}$. In the absence of body forces, the in-plane stresses $\tau_{\alpha\beta}$ must satisfy the equilibrium equations

$$\partial\tau_{\alpha\beta}/\partial y_\beta = 0, \quad \tau_{\beta\alpha} = \tau_{\alpha\beta} \quad \text{on} \quad \mathscr{D}^* \tag{2.5}$$

and for a plane deformation of a homogeneous, isotropic (hyper-elastic) solid

$$W(x_1, x_2) = \Theta(I(x_1, x_2), \; J(x_1, x_2)) \quad \text{for all} \quad (x_1, x_2) \in \mathscr{D}, \tag{2.6}$$

where Θ is the plane-strain elastic potential. Moreover, the governing constitutive relations may now be written as

$$\tau_{\alpha\beta} = \frac{2}{J}\frac{\partial\Theta}{\partial I} F_{\alpha\rho}F_{\beta\rho} + \frac{\partial\Theta}{\partial J}\delta_{\alpha\beta} \quad \text{on} \quad \mathscr{D}. \tag{2.7}$$

The Piola stress-tensor field $\boldsymbol{\sigma}$ associated with the Cauchy stress field $\boldsymbol{\tau}$ is defined by

$$\boldsymbol{\sigma} = J\boldsymbol{\tau}(\mathbf{F}^{-1})^T \quad \text{on} \quad \mathscr{D}, \tag{2.8}$$

where $\mathbf{F}^{-1}$ is the inverse of the deformation-gradient tensor. Consequently, the "actual" stresses $\tau_{\alpha\beta}$ are expressible in terms of the "pseudo-stresses" $\sigma_{\alpha\beta}$ through

$$\tau_{\alpha\beta} = \frac{1}{J}\sigma_{\alpha\rho}F_{\beta\rho} \quad \text{on} \quad \mathscr{D}. \tag{2.9}$$

Also, (2.5), (2.7), in view of (2.8), are found to be equivalent to

$$\partial\sigma_{\alpha\beta}/\partial x_\beta = 0 \quad \text{on} \quad \mathscr{D}, \tag{2.10§}$$

†The initial portion of this resumé is taken from [2] and is included here in order to render the present paper reasonably self-contained.

‡The superscript T indicates transposition.

§Observe that $\boldsymbol{\sigma}$, in contrast to $\boldsymbol{\tau}$, is in general not symmetric.

$$\sigma_{\alpha\beta} = 2\frac{\partial\Theta}{\partial I}F_{\alpha\beta} + \frac{\partial\Theta}{\partial J}\epsilon_{\alpha\rho}\epsilon_{\beta\gamma}F_{\rho\gamma} \quad \text{on} \quad \mathcal{D}. \tag{2.11}$$

For our purposes it is essential to recall two properties of the pseudo-stress field $\boldsymbol{\sigma}$. To this end, suppose first that Γ is a regular arc in $\mathcal{D}$, let Γ^* be the image of Γ under the mapping (2.1), denoting by $\mathbf{n}$ and $\mathbf{n}^*$ the unit normal vectors of Γ and Γ^*, respectively. If s and $\mathbf{t}$ are the Piola and the Cauchy traction-vectors given by

$$s_\alpha = \sigma_{\alpha\beta}n_\beta \quad \text{on} \quad \Gamma, \quad t_\alpha = \tau_{\alpha\beta}n^*_\beta \quad \text{on} \quad \Gamma^*, \tag{2.12}$$

one has

$$s_\alpha = 0 \quad \text{on } \Gamma \text{ if and only if } t_\alpha = 0 \quad \text{on} \quad \Gamma^*. \tag{2.13}$$

Second, if Γ is a simple closed regular curve, which—together with its interior—lies wholly in $\mathcal{D}$, then one has the conservation law†

$$\int_\Gamma (Wn_\alpha - s_\beta \partial u_\beta/\partial x_\alpha)\,\mathrm{d}\mathcal{S} = 0, \tag{2.14}$$

where $\mathbf{n}$ is the unit normal vector of Γ and s the Piola traction-vector on Γ.

The linear theory of plane elastostatic strain is recovered from the corresponding finite theory summarized above upon subjecting the latter to a systematic linearization with respect to the displacement-gradient components $\partial u_\alpha/\partial x_\beta$. Under this linearization the strain-energy density passes over into the familiar quadratic form in the components of infinitesimal strain and becomes fully determinate but for its dependence on two elastic constants. Also, in this transition the distinction between the Cauchy and the Piola stresses disappears so that (2.7), as well as (2.11), merge into the stress-displacement relations (1.4). Finally, the conservation law (2.14) survives the linearization unaltered, its validity within linear elastostatics being at the same time a rigorous consequence of the linear theory.

It is clear from (2.7) that the mechanical response of a homogeneous and isotropic elastic solid to a plane deformation—as far as the in-plane stresses are concerned—is governed entirely by the material response function Θ. We turn now to the particular class of *harmonic* materials, introduced and explored by John[10], for which the two-dimensional elastic potential has the form‡

$$\Theta(I, J) = 2\mu[H(R) - J], \quad R = \sqrt{(I + 2J)}, \quad \mu > 0, \tag{2.15}$$

where μ is a constant, while H is a function defined for all positive real arguments.§ We shall assume further that H has continuous derivatives of all orders on its domain of definition. In the present circumstances the constitutive relations (2.7) become

$$\tau_{\alpha\beta} = 2\mu\left\{\frac{H'(R)}{RJ}F_{\alpha\rho}F_{\beta\rho} + \left[\frac{H'(R)}{R} - 1\right]\delta_{\alpha\beta}\right\}, \tag{2.16}$$ ‖

and (2.11) give rise to

$$\sigma_{\alpha\beta} = 2\mu\left\{\frac{H'(R)}{R}F_{\alpha\beta} + \left[\frac{H'(R)}{R} - 1\right]\epsilon_{\alpha\rho}\epsilon_{\beta\gamma}F_{\rho\gamma}\right\}. \tag{2.17}$$

Equations (2.17), in turn, may be written as

$$\left.\begin{array}{ll} \sigma_{11} = 2\mu(A_2 - F_{22}), & \sigma_{22} = 2\mu(A_2 - F_{11}), \\ \sigma_{12} = 2\mu(A_1 + F_{21}), & \sigma_{21} = 2\mu(-A_1 + F_{12}), \end{array}\right\} \tag{2.18}$$

†See Section 1 of [1] for a sketch of the history of this law.

‡Although in [10] harmonic materials are defined in the context of plane deformations, later on John[11] gave a three-dimensional generalization of this definition.

§Recall that the invariants I and J are both positive.

‖We use primes to indicate differentiation of functions of a single variable.

provided the auxiliary functions A_1 and A_2 are defined by

$$A_1 = \frac{H'(R)}{R}(F_{12} - F_{21}), \quad A_2 = \frac{H'(R)}{R}(F_{11} + F_{22}). \tag{2.19}$$

Also, in view of (2.18), (2.2), and the assumed smoothness of the plane deformation (2.1), the equilibrium equations (2.10) now reduce to the Cauchy–Riemann equations

$$A_{1,1} = A_{2,2}, \quad A_{1,2} = -A_{2,1}, \tag{2.20}$$

so that A_1 and A_2 are conjugate harmonic functions.

When (2.1) is the identity mapping, **F** and **G** are each the idem tensor, so that (2.4) and the second of (2.15) give $I = 2$, $J = 1$ and $R = 2$ for the undeformed state. The requirement that the strain-energy density and the actual stresses vanish in this configuration, by virtue of (2.15), (2.16), yields

$$H(2) = 1, \quad H'(2) = 1. \tag{2.21}$$

Let λ_1^2, λ_2^2 be the principal values of the symmetric, positive-definite deformation tensor **G**, whence λ_1, λ_2 ($\lambda_\alpha > 0$) represent the local principal stretches† of the plane deformation (2.1). Then (2.4) and the second of (2.15) imply

$$I = \lambda_1^2 + \lambda_2^2, \quad J = \lambda_1\lambda_2, \quad R = \lambda_1 + \lambda_2 \tag{2.22}$$

and (2.15), (2.22) furnish

$$\Theta(I, J) = \Omega(\lambda_1, \lambda_2) = 2\mu[H(\lambda_1 + \lambda_2) - \lambda_1\lambda_2], \tag{2.23}$$

where Ω is evidently the strain-energy density of a harmonic material regarded as a function of the principal stretches. It follows from (2.23), (2.21) that the strain-energy density is positive, except in the undeformed state ($\lambda_1 = \lambda_2 = 1$), if and only if

$$H(R) > R^2/4 \quad \text{for all} \quad R > 0, \quad R \neq 2. \tag{2.24}$$

Next, the linearization of the elastic potential (2.15) accompanying the previously described transition to the infinitesimal theory of plane strain reveals that μ is the shear modulus for infinitesimal deformations of a harmonic material, whereas $H''(2)$ is related to Poisson's ratio. In this manner one arrives at

$$H''(2) = \frac{1-\nu}{1-2\nu} > 0. \qquad (2.25)‡$$

Additional restrictions on the response function H arise from physical requirements concerning the response of the material to *pure homogeneous plane deformations*. In this case (2.1) has the form

$$y_\alpha = \lambda_\alpha x_\alpha \quad \text{(no sum)}, \tag{2.26}$$

where λ_1, λ_2 are positive constants and are readily identified as the associated principal stretches. Further, (2.16), in conjunction with (2.2), (2.4) and second of (2.15), permit one to conclude that the actual stresses $\tau_{\alpha\beta}$ induced by the deformation (2.26) are constant, with $\tau_{12} = \tau_{21} = 0$ and

$$\tau_{\alpha\alpha}(\lambda_1, \lambda_2) = 2\mu\left[\frac{\lambda_\alpha H'(\lambda_1 + \lambda_2)}{\lambda_1\lambda_2} - 1\right] \quad \text{(no sum)}. \qquad (2.27)§$$

†Recall that λ_α, for fixed α, corresponds to an elongation or a contraction according as $\lambda_\alpha > 1$ or $\lambda_\alpha < 1$.

‡Thus, as pointed out by John [10], $H''(2) = c_1^2/2c_2^2$, if at present c_1 and c_2 are the speeds of irrotational and equivoluminal waves in linear elastodynamics.

§In the present context we regard the (position-independent) stresses $\tau_{\alpha\beta}$ as functions of the principal stretches.

Consider first the special state of *plane isotropic extension*, characterized by $\lambda_1 = \lambda_2 = \lambda$. Here one draws from (2.27) that

$$\tau_{\alpha\alpha}(\lambda, \lambda) = \tau(\lambda) = 2\mu\left[\frac{H'(2\lambda)}{\lambda} - 1\right] \quad \text{(no sum)}, \quad (0 < \lambda < \infty). \tag{2.28}$$

Evidently, $\tau(\lambda)$ in (2.28) *is monotone increasing with* λ *if and only if*

$$H'(R)/R \text{ is monotone increasing for } 0 < R < \infty, \tag{2.29}$$

and (2.29) will henceforth be assumed to hold on physical grounds.

Next, consider a state of plane-strain *uni-axial tension*—parallel to the x_1-axis—appropriate to a principal stretch $\lambda_1 = \lambda$ and call $\bar{\tau}(\lambda)$ the corresponding normal stress, while denoting by $\lambda_2 = \bar{\lambda}(\lambda)$ the ensuing transverse principal stretch, so that

$$\tau_{11}(\lambda_1, \lambda_2) = \tau_{11}(\lambda, \bar{\lambda}(\lambda)) = \bar{\tau}(\lambda), \quad \tau_{22}(\lambda, \bar{\lambda}(\lambda)) = 0 \quad (0 < \lambda < \infty). \tag{2.30}$$

In view of (2.30), (2.27), and for obvious physical reasons, we shall say that *a harmonic material admits a regular state of uni-axial tension* (in plane strain) *provided for any given* $\lambda > 0$ *there exists a unique, differentiable root* $\lambda_2 = \bar{\lambda}(\lambda)$ *of the equation*

$$H'(\lambda + \lambda_2) = \lambda, \tag{2.31}$$

such that

$$\bar{\lambda}(\lambda) > 0, \quad \bar{\lambda}(\lambda) \quad \text{is monotone decreasing for} \quad 0 < \lambda < \infty. \tag{2.32}$$

We now prove the following theorem: *necessary and sufficient in order that a harmonic material admit a regular state of uni-axial tension in plane strain is that*

$$\text{there exists an } R_o \in (1, 2) \text{ such that } H'(R_o) = 0, \tag{2.33}$$

$$H''(R) > 1 \quad (R_o < R < \infty), \quad H'(R)/R \to 1 \text{ as } R \to \infty. \tag{2.34}$$

In this connection we take for granted that H obeys (2.29), the inequality supplied by (2.25), as well as (2.21).†

To establish the *necessity* of (2.33), (2.34) we show first that

$$1 < \bar{\lambda}(0+) < \infty. \tag{2.35}$$

The left inequality in (2.35) is immediate from $\bar{\lambda}(1) = 1$ and the monotonicity hypothesis in (2.32). To confirm the right inequality in (2.35), note on the basis of (2.29), (2.21) that $H'(R) \to \infty$ as $R \to \infty$. Hence $\bar{\lambda}(0+) = +\infty$ contradicts (2.31) and thus (2.35) must hold true. Now define R_o through

$$R_o = \bar{\lambda}(0+). \tag{2.36}$$

Then, by (2.31), $H'(R_o) = 0$ and hence (2.29), (2.21) now assure that $R_o < 2$, whence (2.33) has been verified.

To see that (2.34) too are necessary insert $\lambda_2 = \bar{\lambda}(\lambda)$ in (2.31) and differentiate the resulting identity with respect to λ to obtain

$$H''(\lambda + \bar{\lambda}(\lambda))[1 + \bar{\lambda}'(\lambda)] = 1 \quad (0 < \lambda < \infty). \tag{2.37}$$

Accordingly, either both factors of the left member in (2.37) are positive or both are negative for all $\lambda < 0$. Since $\bar{\lambda}(1) = 1$ and because of (2.25), it follows that

$$H''(\lambda + \bar{\lambda}(\lambda)) > 0, \quad 1 + \bar{\lambda}'(\lambda) > 0 \quad (0 < \lambda < \infty), \tag{2.38}$$

and (2.37), (2.38), together with the monotonicity of $\bar{\lambda}(\lambda)$ required in (2.32), give

$$\bar{\lambda}'(\lambda) = \frac{1}{H''(\lambda + \bar{\lambda}(\lambda))} - 1 < 0 \quad (0 < \lambda < \infty). \tag{2.39}$$

On the other hand, the argument of H'' in (2.39), on account of the second of the inequalities (2.38), is a monotone increasing function of λ; because of (2.36), this function maps the interval $(0, \infty)$ one-to-one onto (R_o, ∞). Consequently (2.38) and (2.39) enable one to infer the first of (2.34). Finally, from the hypothesis concerning the existence of a root of (2.31) follows

$$\frac{H'(\lambda + \bar{\lambda}(\lambda))}{\lambda + \bar{\lambda}(\lambda)} = \frac{\lambda}{\lambda + \bar{\lambda}(\lambda)} \quad (0 < \lambda < \infty), \tag{2.40}$$

and proceeding to the limit in (2.40) as $\lambda \to \infty$, while bearing (2.32) in mind, one concludes that the second of (2.34) is valid.

Our next objective is to demonstrate the *sufficiency* of (2.33) and (2.34). For this purpose we observe by recourse to the first of (2.34) that H' is a monotone increasing function on $[R_o, \infty)$. Also, since $H'(R_o) = 0$ and $H'(\infty) = +\infty$, as noted earlier, H' maps the interval $[R_o, \infty)$ one-to-one onto $[0, \infty)$. Thus and because of the assumed smoothness of H, the function H' has a unique differentiable inverse on $(0, \infty)$, which we denote by Λ, whence

$$H'(\Lambda(\lambda)) = \lambda \quad (0 < \lambda < \infty). \tag{2.41}$$

Now define a function $\bar{\lambda}$ on $(0, \infty)$ by means of

$$\bar{\lambda}(\lambda) = \Lambda(\lambda) - \lambda \quad (0 < \lambda < \infty). \tag{2.42}$$

Evidently, $\lambda_2 = \bar{\lambda}(\lambda)$ is a differentiable root of (2.31) for $0 < \lambda < \infty$. Moreover, from (2.29) and the second of (2.34) one draws $H'(R)/R < 1$ for all $R > R_o$, so that

$$\frac{H'(\Lambda(\lambda))}{\Lambda(\lambda)} = \frac{\lambda}{\Lambda(\lambda)} < 1 \quad (0 < \lambda < \infty), \tag{2.43}$$

and hence $\bar{\lambda}(\lambda)$, defined in (2.42), is positive for all $\lambda > 0$, as required. In addition, the first of (2.34) and the properties of $\bar{\lambda}(\lambda)$ already established entitle us to assert (2.39) once again; consequently, $\bar{\lambda}$ is monotone decreasing on $(0, \infty)$. Clearly, $\lambda_2 = \bar{\lambda}(\lambda)$ is the only root of (2.31) such that $\lambda + \bar{\lambda}(\lambda) > R_o$ since H' is monotone increasing on (R_o, ∞). At the same time there cannot exist $\lambda > 0$ such that $\lambda_2 = \lambda_*(\lambda) > 0$ is a root of (2.31) with $\lambda + \lambda_*(\lambda) \leq R_o$ because the latter inequality implies $H'(\lambda + \lambda_*(\lambda)) \leq 0$, in view of (2.29), (2.33). This confirms the uniqueness of the root constructed above and completes the proof of the theorem.

From here on we confine our attention to harmonic materials that admit a regular state of uni-axial tension in plane strain. It is clear from (2.27), (2.30) that for such materials

$$\bar{\tau}(\lambda) = 2\mu\left[\frac{\lambda}{\bar{\lambda}(\lambda)} - 1\right] \quad (0 < \lambda < \infty), \tag{2.44}$$

and since $\bar{\lambda}(\lambda)$ is decreasing and positive, $\bar{\tau}(\lambda)$ *is monotone increasing* for $0 < \lambda < \infty$; further, (2.33), (2.36) together with the stress-stretch relation (2.44) for uni-axial tension and its analogue (2.28) for isotropic extension furnish

$$\bar{\tau}(0+) = -2\mu, \quad 1 < \bar{\lambda}(0+) = R_o < 2, \quad \tau(\lambda) \to 2\mu \quad \text{as} \quad \lambda \to \infty, \tag{2.45}$$

the physical interpretation of which is immediate.

It should be remarked that the foregoing conclusions concerning the behavior of harmonic materials under isotropic extension and uni-axial tension are implicit in John's[10] discussion of such plane homogeneous deformations (see Sections 2.1, 2.2 in [10]). For our purposes it is essential to introduce assumptions regarding the asymptotic structure of the response function $H(R)$, as $R \to \infty$, beyond those contained in (2.29), (2.34). From the second of (2.34) follows

$$H(R) \sim \frac{R^2}{2}, \quad H'(R) \sim R \quad \text{as} \quad R \to \infty. \tag{2.46}$$

Referring to the limit as $R \to \infty$, we stipulate in addition that

$$H(R) \sim \frac{R^2}{2} - \alpha R^n, \quad H'(R) \sim R - \alpha n R^{n-1}, \quad H''(R) \sim 1 - \alpha n(n-1) R^{n-2}, \tag{2.47}$$

where α and n are material constants subject to

$$\alpha \neq 0, \quad n \neq 0, \quad n < 2. \tag{2.48}$$

Evidently (2.47), (2.48) are consistent with (2.34) only if

$$\alpha n > 0, \quad n \leq 1. \tag{2.49}$$

Combining (2.31) with the second of (2.47) and invoking (2.44), (2.49), as well as the boundedness of $\bar{\lambda}(\lambda)$ for all $\lambda > 0$, one arrives at the subsequent asymptotic relations governing uni-axial tension in the presence of large axial extensions:

$$\bar{\tau}(\lambda) \sim \frac{2\mu}{\alpha n} \lambda^{2-n}, \quad \bar{\lambda}(\lambda) \sim \alpha n \lambda^{n-1} \quad \text{as} \quad \lambda \to \infty. \tag{2.50}$$

In particular, when $n = 1$, as we shall later on assume to be the case, one has

$$\bar{\tau}(\lambda) \sim \frac{2\mu}{\alpha} \lambda, \quad \bar{\lambda}(\lambda) \sim \alpha \quad \text{as} \quad \lambda \to \infty, \quad 0 < \alpha < 1. \tag{2.51}$$

Hence, in this instance, the asymptotic stress-stretch relation is linear† and α is the limiting value of the transverse stretch $\bar{\lambda}(\lambda)$ as $\lambda \to \infty$. While $\alpha > 0$ is implied by (2.49) with $n = 1$, the inequality $\alpha < 1$ follows from (2.32), according to which $\bar{\lambda}(\lambda) < 1$ (contraction) for all $\lambda > 1$.

We mention in passing that a harmonic material which admits a regular state of uni-axial tension is readily found to possess physically acceptable response characteristics in *simple shear.* At this stage we turn from the discussion of *special* plane deformations of harmonic materials to certain requirements that stem from restrictions commonly imposed on the response of an elastic material to *all* deformations.

Thus Coleman and Noll[12] argued on thermostatic grounds that the strain-energy density must be a convex function of the principal stretches. This condition in the present context necessitates that the matrix

$$[\partial^2 \Omega / \partial \lambda_\alpha \partial \lambda_\beta] \quad \text{is positive semi-definite,} \tag{2.52}$$

where Ω is the function defined in (2.23). On the other hand, (2.52) is seen to hold if and only if

$$H''(R) \geq \frac{1}{2} \quad \text{for all} \quad R > 0, \tag{2.53}$$

while, conversely, the strong inequality in (2.53) suffices to insure the convexity of Ω. Hence the

†It is easily confirmed that the slope of the stress-stretch curve as $\lambda \to \infty$ is larger than its slope at $\lambda = 1$.

Coleman-Noll condition is satisfied for plane deformations of a harmonic material provided the first of (2.34) is supplemented by

$$H''(R) > \frac{1}{2} \quad (0 < R < R_o). \tag{2.54}$$

Next, according to a general postulate of Baker and Ericksen[13], which rests on physical plausibility, the axis of the larger principal stress must coincide with the axis of the larger principal stretch in any plane pure homogeneous deformation other than isotropic extension, i.e.

$$(\tau_{11} - \tau_{22})(\lambda_1 - \lambda_2) > 0 \quad \text{for all} \quad \lambda_\alpha > 0, \quad \lambda_1 \neq \lambda_2. \tag{2.55}$$

As is clear from (2.27), a necessary and sufficient condition that this inequality hold true for a harmonic material is that

$$H'(R) > 0 \quad \text{for all} \quad R > 0. \tag{2.56}$$

But (2.29), (2.33), which arose from the demand that the material exhibit a physically reasonable behavior in isotropic extension and uni-axial tension, imply

$$H'(R) < 0 \quad (0 < R < R_o), \quad H'(R) > 0 \quad (R_o < R < \infty). \tag{2.57}$$

Consequently, the relevant Baker–Ericksen inequality (2.55) is satisfied for the class of harmonic materials admitted here only when $\lambda_1 + \lambda_2 > R_o$ and is violated at all points (λ_1, λ_2) of the principal-stretch plane for which

$$\lambda_1 + \lambda_2 \leq R_o = \bar{\lambda}(0+), \quad \lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_1 \neq \lambda_2, \tag{2.58}$$

as was observed already by John[10]. Since $1 < R_o < 2$, the domain of validity of (2.55) contains an entire neighborhood of the undeformed state ($\lambda_1 = \lambda_2 = 1$), as well as any region of the (λ_1, λ_2)-plane in which either $\lambda_1 > 2$ or $\lambda_2 > 2$. Note further, by recourse to (2.27), (2.56), that both principal stresses are necessarily compressive whenever (2.55) fails to hold.

We return here briefly to the field equations and constitutive relations governing plane deformations of a harmonic material in order to cast these equations into a form especially convenient for our purpose. This may be accomplished by introducing certain complex combinations of some of the real-valued functions entering the two-dimensional theory reviewed earlier.

To begin with, the plane deformation (2.1) may be described in terms of the single complex-valued function w defined by

$$y_1 + iy_2 = w(x_1, x_2) = \hat{y}_1(x_1, x_2) + i\hat{y}_2(x_1, x_2) \quad \text{for all} \quad (x_1, x_2) \in \mathcal{D}. \tag{2.59}$$

Then (2.2) in conjunction with (2.4) and the second of (2.15) lead to the representations

$$I = |w_{,1}|^2 + |w_{,2}|^2, \quad J = -Im\{w_{,1}\bar{w}_{,2}\}, \quad R = |w_{,2} + iw_{,1}|. \qquad (2.60)\dagger$$

Next set

$$A(x_1, x_2) = A_1(x_1, x_2) + iA_2(x_1, x_2) \quad \text{for all} \quad (x_1, x_2) \in \mathcal{D}. \tag{2.61}$$

where A_1, A_2 are the auxiliary functions originally introduced in (2.19). From (2.61) and (2.19) one draws with the aid of (2.59) and (2.2) that

$$A = \frac{H'(R)}{R}(w_{,2} + iw_{,1}), \tag{2.62}$$

†As before, subscripts preceded by a comma indicate partial differentiation with respect to the corresponding (material) cartesian coordinate; $\bar{w}$ is the complex conjugate of w.

whereas the constitutive relations (2.18) for the Piola stresses may now be written as

$$\sigma_{12}+i\sigma_{22}=2\mu(A-iw_{,1}),\quad \sigma_{11}+i\sigma_{21}=2\mu i(-A+w_{,2}). \tag{2.63}$$

Further, the constitutive relations (2.16) for the Cauchy stresses, after some elementary manipulation, are found to be equivalent to

$$\tau_{11}+\tau_{22}=2\mu\left[\frac{RH'(R)}{J}-2\right],\quad \tau_{11}-\tau_{22}+2i\tau_{12}=2\mu\frac{H'(R)}{JR}(w_{,1}^2+w_{,2}^2). \tag{2.64}$$

Finally, the equilibrium conditions (2.20) at present take the form

$$A_{,1}+iA_{,2}=0. \tag{2.65}$$

Accordingly A, when regarded as a function of the complex variable $z=x_1+ix_2$, is analytic on the domain $\mathscr{D}$, as is also apparent directly from (2.61) and (2.20).

3. ASYMPTOTIC TREATMENT OF THE MIXED HALF-PLANE PROBLEM FOR FINITE PLANE STRAIN OF A HARMONIC MATERIAL

We proceed now to the analogue in the theory of harmonic materials of the mixed half-plane problem that was discussed in Section 1 within the framework of the linearized equilibrium theory of plane strain. Taking for granted the existence of solutions in the large to the corresponding nonlinear mixed problem, we explore here their detailed asymptotic behavior near the origin under certain restrictive assumptions to be spelled out shortly.

An expedient characterization of the global solutions we wish to investigate asymptotically may be based on the complex version of the equations governing plane deformations of a harmonic material, summarized at the end of Section 2. With this objective in mind we let (r,θ) once more be the polar coordinates appearing in (1.1) and $\mathscr{R}$ the (punctured) half-plane defined in (1.2). We suppose there exists a plane deformation (2.59) with a complex spatial coordinate w, suitably smooth on $\mathscr{R}$, such that the function A generated by w through (2.62) and the last of (2.60), satisfies the equilibrium requirement (2.65) on the interior $\mathring{\mathscr{R}}$. Furthermore, we suppose that w and the Piola stresses associated with w by means of (2.63), (2.62) satisfy the boundary conditions

$$\sigma_{12}+i\sigma_{22}=0\quad\text{at}\quad\theta=0,\quad w=x_1\quad\text{at}\quad\theta=\pi. \tag{3.1}$$

The first of (3.1), because of the result cited in (2.13) assures that the deformation-image of the leg $\theta=0$ of the boundary is free of tractions; the second of (3.1) asserts the fixity of the leg $\theta=\pi$.

From here on we shall consistently regard w and all associated fields as functions of the material *polar* coordinates† (r,θ). Equations (2.60) referred to polar coordinates become

$$I=\left|\frac{\partial w}{\partial r}\right|^2+\frac{1}{r^2}\left|\frac{\partial w}{\partial\theta}\right|^2,\quad J=-Im\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial\bar{w}}{\partial\theta}\right\},\quad R=\left|\frac{1}{r}\frac{\partial w}{\partial\theta}+i\frac{\partial w}{\partial r}\right|. \tag{3.2}$$

The polar equivalent of (2.62), in turn, is found to yield

$$A=\frac{H'(R)}{R}\left(\frac{1}{r}\frac{\partial w}{\partial\theta}+i\frac{\partial w}{\partial r}\right)\exp(-i\theta)\quad(0<r<\infty,0<\theta<\pi) \tag{3.3}$$

and the equilibrium condition (2.65), which requires the analyticity of A, now gives

$$\frac{\partial A}{\partial r}+\frac{i}{r}\frac{\partial A}{\partial\theta}=0\quad(0<r<\infty,0<\theta<\pi), \tag{3.4}$$

while the boundary conditions (3.1), in view of (1.1) and the first of (2.63), may at present be

†In order to avoid unduly cumbersome notation we shall retain the same symbols employed previously to denote these fields as functions of the material cartesian coordinates.

written as

$$A(r,0)-i\frac{\partial}{\partial r}w(r,0)=0,\quad w(r,\pi)=-r\quad (0<r<\infty). \tag{3.5}$$

Evidently, (3.3), (3.4), supported by the last of (3.2) and accompanied by the boundary conditions (3.5), cannot be expected to determine w uniquely for a given response function H. Indeed, even the counterpart in the linearized theory of the foregoing mixed half-plane problem has an infinity of solutions, as was recalled in Section 1. At this stage we confine our attention to solutions w that admit an asymptotic representation

$$w(r,\theta)=r^m U(\theta)+o(r^m)\quad \text{as}\quad r\to 0\quad (0\le\theta\le\pi),\quad 0<m<1, \tag{3.6}$$

where m is a real constant† and U a complex-valued function that is at least twice continuously differentiable on $[0,\pi]$ and fails to vanish identically. Further, (3.6) is to hold uniformly with respect to θ for $0\le\theta\le\pi$ and is understood to include the additional hypothesis that the first and second partial derivatives of w admit strictly analogous asymptotic representations, obtainable from (3.6) by formal differentiation.

Since m is restricted to positive values in (3.6), the latter implies that the spatial coordinates stay bounded as $r\to 0$, the origin remaining fixed. In contrast, because of the restriction $m<1$ in (3.6), the companion assumptions regarding the asymptotic behavior of the derivatives of w imply that not all deformation-gradient components remain bounded at the origin. In view of the assumed asymptotic structure of w and of its derivatives, the third of (3.2) furnishes

$$R=r^{m-1}|U'+imU|+o(r^{m-1})\quad \text{as}\quad r\to 0\quad (0\le\theta\le\pi), \tag{3.7}$$

where the prime indicates differentiation with respect to θ. Also, the dominant term in (3.7) cannot vanish identically: otherwise (3.6) and the second of (3.2) give

$$w\sim \mathrm{K}r^m\exp(-im\theta),\quad J\sim -m^2|\mathrm{K}|^2r^{2(m-1)}, \tag{3.8‡}$$

$\mathrm{K}\neq 0$ being a complex constant, which contradicts the required positivity of J. Consequently

$$R(r,\theta)\to\infty\quad \text{as}\quad r\to 0\quad (0\le\theta\le\pi), \tag{3.9}$$

and thus at least one of the principal stretches becomes arbitrarily large as $r\to 0$. We observe incidentally that according to (3.9) the values of R arising at material points sufficiently close to the origin lie within the range of validity of the Baker–Ericksen inequality§ (2.55).

We now attempt to find the exponent m and the unknown function U appearing in (3.6). From (3.6), (3.3), (3.9) and the asymptotic behavior, as $R\to\infty$, of $H'(R)$ stated in (2.46), follows

$$A\sim r^{m-1}(U'+imU)\exp(-i\theta). \tag{3.10}$$

Entering the equilibrium equation (3.4) and the boundary conditions (3.5) with (3.10) and (3.6), one finds that (3.4), (3.5) demand

$$U''+m^2U=0\quad \text{on}\quad [0,\pi],\quad U'(0)=0,\quad U(\pi)=0. \tag{3.11}$$

Equations (3.11) constitute a linear eigenvalue problem for U with $m\in(0,1)$ as the eigenvalue parameter. The solution of this problem is clearly given by

$$m=\frac{1}{2},\quad U(\theta)=a\cos(\theta/2)\quad (0\le\theta\le\pi), \tag{3.12}$$

†We shall consider later on the generalization of (3.6) to complex values of the parameter m. See the remark following (3.13).

‡From here on all asymptotic equalities, unless otherwise qualified, refer to the limit as $r\to 0$.

§See (2.56), (2.57).

in which a is an arbitrary complex constant (amplitude parameter). Combining (3.12) with (3.6) we arrive at the lowest-order asymptotic solution of the problem under consideration in the form

$$w \sim ar^{1/2} \cos(\theta/2). \tag{3.13}$$

It is easily confirmed that one obtains precisely the same result if complex values of the exponent m are admitted in (3.6), provided the inequality appearing there is replaced by $0 < Re\{m\} < 1$.

The lowest-order approximation to w furnished by the right-hand member of (3.13) is severely degenerate since its Jacobian determinant vanishes identically, as can be seen at once with the aid of the second of (3.2). In fact (3.13) yields merely the weak estimate $J = o(r^{-1})$ as $r \to 0$ and hence leaves indeterminate the actual stresses associated with w through (2.64). It is therefore essential to establish at least a second-order approximation to w. This leads us to replace (3.6) by the two-term asymptotic representation

$$w \sim r^{1/2} U(\theta) + r^s V(\theta) \quad (0 \le \theta \le \pi), \quad s > 1/2, \tag{3.14}$$

where U is the function given in (3.12), while s is an as yet unknown real parameter and V another initially unknown complex-valued function that has the properties demanded of U in connection with (3.6). Moreover, we stipulate that (3.14), together with the asymptotic equalities resulting from one or two formal partial differentiations of (3.14), hold uniformly in θ for $0 \le \theta \le \pi$. In addition we assume that the response function H obeys (2.47) with $n = 1$ whence, in particular,

$$\frac{H'(R)}{R} \sim 1 - \frac{\alpha}{R} \quad \text{as} \quad R \to \infty, \quad 0 < \alpha < 1. \tag{3.15}†$$

Our current aim is to find V and the smallest value of s consistent with (3.14), (3.15) and (3.3), (3.4), (3.5).

From (3.12), (3.14), (3.15) and (3.3) one draws

$$A \sim \frac{ia}{2} r^{-1/2} \exp(-i\theta/2) + r^{s-1}(V' + isV) \exp(-i\theta) - \frac{i\alpha a}{|a|} \exp(-i\theta/2). \tag{3.16}$$

Suppose first $1/2 < s < 1$. In this event the second term in (3.16) dominates over the third and thus

$$A \sim \frac{ia}{2} r^{-1/2} \exp(-i\theta/2) + r^{s-1}(V' + isV) \exp(-i\theta). \tag{3.17}$$

But inserting (3.14), (3.17) in the equilibrium equation (3.4) and the boundary condition (3.5), one is led to (3.11) with U and m replaced by V and s, respectively. Since this homogeneous boundary-value problem fails to admit a nontrivial solution when $1/2 < s < 1$, one concludes that $s \ge 1$.

Consider next (3.14), (3.16) with $s = 1$. Then (3.4), (3.5), (3.12) are easily seen to imply that

$$V'' + V = \frac{\alpha a}{2|a|} \exp(i\theta/2) \quad \text{on} \quad [0, \pi], \quad V'(0) = \frac{i\alpha a}{|a|}, \quad V(\pi) = -1. \tag{3.18}$$

The boundary-value problem (3.18) has a solution, which is unique and given by

$$V(\theta) = \cos\theta + \frac{2i\alpha a}{3|a|}[\cos\theta + \sin\theta - i\exp(i\theta/2)] \quad (0 \le \theta \le \pi), \tag{3.19}$$

so that there is no need to explore the case $s > 1$. In view of (3.19) and (3.14) with $s = 1$,

†Recall (2.51) for the inequality satisfied by the material parameter α when $n = 1$.

the second-order asymptotic approximation to w reads:

$$w \sim ar^{1/2}\cos\frac{\theta}{2} + r\left[\cos\theta + \frac{2i\alpha a}{3|a|}(\cos\theta + \sin\theta - i\exp(i\theta/2)\right]. \tag{3.20}$$

On setting

$$a = a_1 + ia_2 \neq 0 \quad (a_1, a_2 \quad \text{real constants}), \tag{3.21}$$

one infers from (3.20) and the second of (3.2) that

$$J \sim r^{-1/2}p(\theta), \tag{3.22}$$

provided

$$p(\theta) = \frac{a_2}{2}\sin\frac{\theta}{2} + \frac{\alpha|a|}{3}\left(\frac{3}{4} - \frac{\cos\theta}{4} + \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right) \quad (0 \leq \theta \leq \pi). \tag{3.23}$$

Clearly, a necessary condition that J be positive is that $p \geq 0$ on $[0, \pi]$ and this requirement is fulfilled if and only if

$$a_2 \geq 0. \tag{3.24}$$

Note further that, α and $|a|$ being necessarily positive,

$$p(\theta) > 0 \quad (0 \leq \theta < \pi), \quad p(\pi) = \frac{a_2}{2}, \quad p(0) = \frac{\alpha|a|}{2} > 0, \tag{3.25}$$

whence the estimate (3.22), (3.23) is nondegenerate for $0 \leq \theta \leq \pi$ if $a_2 > 0$. The same is true for $0 \leq \theta < \pi$ when $a_2 = 0$, but (3.22) supplies merely the weak estimate

$$J(r, \pi) = o(r^{-1/2}) \quad \text{if} \quad a_2 = 0, \tag{3.26}$$

which leaves the actual stresses (2.64) indeterminate at $\theta = \pi$. It is this unfortunate circumstance that compels us to pursue a third-order asymptotic approximation of w.

In the preceding second-order analysis we assumed that the response function H conforms to (2.47) with $n = 1$, which led to (3.15). The latter estimate is, however, insufficient for a determination of w to third order. We therefore refine our previous asymptotic material assumptions by requiring that

$$\frac{H'(R)}{R} \sim 1 - \frac{\alpha}{R} - \frac{\beta}{R^2} \quad \text{as} \quad R \to \infty, \quad 0 < \alpha < 1, \quad \beta \geq 0. \tag{3.27}$$

The condition that the new material parameter β be non-negative is necessary if (3.27) is to be formally differentiable, because of the first of (2.34).

We now replace (3.14) by the three-term representation

$$w \sim r^{1/2}U(\theta) + rV(\theta) + r^t \log r S(\theta) + r^t T(\theta) \quad (0 \leq \theta \leq \pi) \quad t > 1, \tag{3.28}$$

where U and V are the functions given by (3.12), (3.19), taking for granted that (3.28) and the unknown (complex-valued) functions S, T satisfy hypotheses strictly parallel to those accompanying (3.6) and (3.14). In particular, neither S nor T is permitted to vanish identically. Our aim is to find S and T, as well as the smallest value of the exponent t consistent with (3.28), (3.27) and (3.3), (3.4), (3.5).

The need for the logarithmic term in (3.28) stems from the fact that in its absence the inhomogeneous boundary-value problem emerging for T is found to have no solution. Further, when $1 < t < 3/2$ one encounters the homogeneous boundary-value problem (3.11) with U and m replaced by S and t, respectively, so that $S = 0$ on $[0, \pi]$ for this range of the parameter t. Hence $t \geq 3/2$. We show next that $t = 3/2$ indeed leads to the desired third-order solution for w.

From (3.28) with $t = 3/2$, (3.27), the earlier results for U and V, as well as (3.3), (3.4), (3.5), one infers after fairly elaborate elementary computations that

$$S'' + \frac{9}{4}S = 0 \quad \text{on} \quad [0, \pi], \quad S'(0) = 0, \quad S(\pi) = 0, \tag{3.29}$$

$$T'' + \frac{9}{4}T = -3S + \left[\frac{i}{2}Q - Q'\right]\exp(i\theta) \quad \text{on} \quad [0, \pi], \quad T'(0) = -Q(0), \quad T(\pi) = 0, \tag{3.30}$$

in which Q is the auxiliary function defined by

$$Q(\theta) = \frac{2\alpha}{3|a|^2}[2(1-i)\alpha a - 3i|a|] + \frac{2ia}{3|a|^3}\left[(3\alpha a_1 + 2\alpha^2|a| - 3\beta|a|)\cos\frac{\theta}{2}\right.$$
$$\left. + (3\alpha a_2 - 2\alpha^2|a| + 3i\beta|a|)\sin\frac{\theta}{2}\right] \quad (0 \le \theta \le \pi). \tag{3.31}$$

Here a is the complex amplitude parameter appearing in the lowest-order solution (3.13), while a_1 and a_2 are again the real and imaginary parts of a.

The complete solution of the problem (3.29) is

$$S(\theta) = k\cos(3\theta/2) \quad (0 \le \theta \le \pi), \tag{3.32}$$

where k is an as yet arbitrary constant, which is to be determined so as to render the ensuing boundary-value problem (3.30) for T solvable. Multiplying the differential equation in (3.30) by $\cos(3\theta/2)$ and integrating the resulting identity from $\theta = 0$ to $\theta = \pi$, one draws–upon integration by parts and use of the boundary conditions in (3.30):

$$k = \frac{i}{\pi}\int_0^\pi Q(\theta)\exp(5i\theta/2)\,d\theta. \tag{3.33}$$

After substitution for S from (3.32) into (3.30) one easily deduces the complete solution for T, which admits the representation

$$T(\theta) = \tilde{a}\cos(3\theta/2) + \frac{ik}{6}[(6\pi + 2i - 3\theta)\exp(-3i\theta/2)$$
$$+ (2i + 3\theta)\exp(3i\theta/2)] + \exp(-3i\theta/2)\int_\theta^\pi Q(\varphi)\exp(5i\varphi/2)\,d\varphi \quad (0 \le \theta \le \pi), \tag{3.34}$$

where $\tilde{a}$ is an arbitrary complex constant, which is left undetermined by the third-order analysis; note that the solutions for S and T involve through Q the original amplitude parameter a. In view of (3.31), the integrals in (3.33), (3.34) can be evaluated in closed elementary form, but in the interest of brevity we omit the explicit results thus obtained, which are not essential to our present purpose.

Equation (3.28) with $t = 3/2$ and (3.32), (3.33), (3.34) furnish the third-order asymptotic approximation to the complex spatial coordinate w. Employing this approximation in conjunction with the second of (3.2), one arrives at the subsequent estimate for the Jacobian determinant:

$$J(r, \theta) = r^{-1/2}p(\theta) + q(\theta)\log r + f(\theta) + o(1) \quad \text{as} \quad r \to 0 \quad (0 \le \theta \le \pi). \tag{3.35}$$

Here p is the function previously introduced in (3.23),

$$q(\theta) = \frac{\alpha}{15\pi|a|}(4\alpha|a| + 3a_1 - 3a_2)\sin\theta \quad (0 \le \theta \le \pi), \tag{3.36}$$

whereas f, which is a somewhat more elaborate elementary function—also depending on the parameters α, β, and a—obeys

$$f(\pi)=\frac{\alpha}{3|a|}(2a_1+a_2). \tag{3.37}$$

From (3.35), (3.25), (3.36), (3.37) follows

$$J(r,\pi)=\frac{a_2}{2}r^{-1/2}+\frac{\alpha}{3|a|}(2a_1+a_2)+o(1) \quad \text{as} \quad r\to 0. \tag{3.38}$$

Since J must be positive and $a\neq 0$, one has:

$$a_2\geq 0; \quad a=ia_2, \quad a_2>0 \quad \text{if} \quad a_1=0; \quad a=a_1>0 \quad \text{if} \quad a_2=0. \tag{3.39}$$

Thus, in particular,

$$J(r,\pi)=\frac{2\alpha}{3}+o(1) \quad \text{if} \quad a_2=0. \tag{3.40}$$

Equation (3.40) supersedes the inadequate estimate (3.26) deduced from the second-order solution for w. It should be remarked that (3.38) can be inferred directly from (3.28) with $t=3/2$ and the boundary conditions in (3.29), (3.30) by recourse to the second of (3.2), if the existence of a solution to (3.29), (3.30) is taken for granted. Finally, we observe that the material constant β does not enter (3.38).

The material assumptions underlying the preceding results are contained entirely in (2.15) and (3.27). Whereas (2.15) asserts that the elastic material considered is of harmonic type, (3.27) describes its specific behavior in the presence of large extensional deformations. It should be noted that the results deduced in this section remain valid even if the material is merely *asymptotically* harmonic in the sense that its plane-strain elastic potential obeys

$$\Theta(I,J)\sim 2\mu\left[\frac{R^2}{2}-\alpha R-\beta\log R-J\right] \quad \text{as} \quad R\to\infty, \quad R=\sqrt{(I+2J)}, \quad \mu>0, \quad 0<\alpha<1, \quad \beta\geq 0. \tag{3.41}$$

Such an assumption would be analogous to the asymptotic material hypothesis upon which the analysis of the crack-problem reported in [1, 2] was based.

4. ASYMPTOTIC RESULTS FOR THE DEFORMATION AND STRESSES. DISCUSSION

In this section we first deduce from the results for w and J established in Section 3 asymptotic approximations, to the relevant orders, for the (real-valued) spatial coordinates y_α and actual stresses $\tau_{\alpha\beta}$.

Decomposing the second-order approximation (3.20) to w into its real and imaginary parts and bearing (2.59) in mind, we obtain

$$\left.\begin{aligned}\hat{y}_1(r,\theta)&\sim a_1r^{1/2}\cos\frac{\theta}{2}+r\left\{\cos\theta+\frac{2\alpha}{3|a|}\left[a_1\cos\frac{\theta}{2}-a_2\left(\sin\theta+\cos\theta+\sin\frac{\theta}{2}\right)\right]\right\},\\ \hat{y}_2(r,\theta)&\sim a_2r^{1/2}\cos\frac{\theta}{2}+\frac{2\alpha}{3|a|}r\left[a_1\left(\sin\theta+\cos\theta+\sin\frac{\theta}{2}\right)+a_2\cos\frac{\theta}{2}\right].\end{aligned}\right\} \tag{4.1}$$

Note that (4.1) is consistent with the boundary conditions at the leg $\theta=\pi$ of the boundary, which is to remain fixed. Of particular interest is the deformation-image of the free portion of the boundary at $\theta=0$, and (4.1) give:

$$\left.\begin{aligned}\hat{y}_1(r,0)&\sim a_1r^{1/2}+\left[1+\frac{2\alpha}{3|a|}(a_1-a_2)\right]r,\\ \hat{y}_2(r,0)&\sim a_2r^{1/2}+\frac{2\alpha}{3|a|}(a_1+a_2)\,r.\end{aligned}\right\} \tag{4.2}$$

If in particular $a_1 = 0$ or $a_2 = 0$, one has the subsequent asymptotic results.

$$\text{For} \quad a_1 = 0,\ a_2 > 0\text{:} \quad \hat{y}_1(r,0) \sim \left(1 - \frac{2\alpha}{3}\right) r, \quad \hat{y}_2(r,0) \sim a_2 r^{1/2} + \frac{2\alpha r}{3}. \tag{4.3}$$

$$\text{For} \quad a_2 = 0,\ a_1 > 0\text{:} \quad \hat{y}_1(r,0) \sim a_1 r^{1/2} + \left(1 + \frac{2\alpha}{3}\right) r, \quad \hat{y}_2(r,0) \sim \frac{2\alpha r}{3}. \tag{4.4}$$

In order to facilitate the computation of the actual stresses to dominant order we first refer the right-hand members in (2.64) to polar coordinates and thus arrive at

$$\left.\begin{aligned} \tau_{11} + \tau_{22} &= 2\mu\left[\frac{RH'(R)}{J} - 2\right], \\ \tau_{11} - \tau_{22} + 2i\tau_{12} &= 2\mu\frac{H'(R)}{JR}\left[\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2\right], \end{aligned}\right\} \tag{4.5}$$

where J and R are related to w through the last two of (3.2). From (2.46), (3.9), (3.13), the last of (3.2) and (3.22) one draws

$$H'(R)/R \sim 1, \quad w \sim ar^{1/2}\cos\frac{\theta}{2}, \quad R \sim \frac{|a|}{2} r^{-1/2}, \quad J \sim p(\theta) r^{-1/2}, \tag{4.6}$$

the estimate for J being non-degenerate for $0 \le \theta \le \pi$, provided† $a_2 \neq 0$ and hence $a_2 > 0$. Combining (4.6) with (4.5) one obtains easily

$$\tau_{\alpha\beta}(r,\theta) \sim \frac{\mu a_\alpha a_\beta}{2p(\theta)} r^{-1/2} \quad (0 \le \theta \le \pi) \quad \text{if} \quad a_2 > 0, \tag{4.7}$$

in which $p(\theta)$ is given by (3.23). Note that (4.7) supplies all three stresses $\tau_{\alpha\beta}$ to dominant order in case $a_2 > 0$, $a_1 \neq 0$, but yields merely

$$\tau_{11} = o(r^{-1/2}), \quad \tau_{22} \sim \frac{\mu a_2^2}{2p(\theta)} r^{-1/2}, \quad \tau_{12} = o(r^{-1/2}) \quad \text{if} \quad a_1 = 0, \quad a_2 > 0 \tag{4.8}$$

and accordingly leads only to weak estimates for τ_{11} and τ_{12} in this instance.

In view of the preceding remarks and because of (3.39), a complete knowledge of the dominant asymptotic behavior of all actual stresses requires, in addition to (4.7), non-degenerate estimates to leading order for $\tau_{\alpha\beta}$ when either $a_1 = 0$ and $a_2 > 0$ or $a_2 = 0$ and $a_1 > 0$. Such estimates are deducible with the aid of the third-order solution for w established in Section 3 and the refined asymptotic material assumption (3.27) underlying this solution. Since the computations here involved are extremely laborious we shall supply the missing estimates exclusively at $\theta = \pi$, i.e. at the fixed leg of the boundary, where the stresses are of particular physical interest. The corresponding results may be summarized as follows.

For $a_1 = 0$, $a_2 > 0$:

$$\tau_{11}(r,\pi) \sim \frac{4\mu}{a_2}\left[\left(1 - \frac{\alpha}{2}\right)^2 + \frac{7\alpha^2}{36}\right] r^{1/2}, \quad \tau_{22}(r,\pi) \sim \mu a_2 r^{-1/2}, \quad \tau_{12}(r,\pi) \sim -\frac{4\mu\alpha}{3}. \tag{4.9}$$

For $a_2 = 0$, $a_1 > 0$:

$$\tau_{11}(r,\pi) \sim \frac{3\mu a_1^2}{4\alpha} r^{-1}, \quad \tau_{22}(r,\pi) \sim \frac{4\mu\alpha}{3}, \quad \tau_{12}(r,\pi) \sim \mu a_1 r^{-1/2}. \tag{4.10}$$

It should be mentioned that the first estimate in (4.9), as well as the second in (4.10), elude a

†Recall (3.24), (3.25).

computation relying on (4.5); these two results were derived by applying (2.62), (2.63) and (2.2), (2.9) to the third-order solution for w. Observe that the second of (4.9) is in agreement with the second of (4.8) because $p(\pi) = a_2/2$ according to (3.25). Although the third-order solution† for w involves the material parameter β, the results in (4.9), (4.10), which are confined to $\theta = \pi$, do not depend on β. To these conclusions regarding the dominant asymptotic behavior of the stresses $\tau_{\alpha\beta}$ we adjoin the subsequent estimate for the strain-energy density W, which may be inferred at once from (2.6), (2.15) together with (2.46), (3.2) and (3.13):

$$W(r, \theta) \sim \frac{\mu}{4}(a_1^2 + a_2^2)\, r^{-1} \quad (0 \le \theta \le \pi). \tag{4.11}$$

The foregoing results characterize the field behavior possible in the vicinity of a point that separates a fixed from a collinear free boundary-segment of an elastic body (finite or otherwise) within the nonlinear equilibrium theory of plane strain for harmonic materials and within the additional assumptions underlying our asymptotic analysis. These assumptions, it should be recalled, include the limitation to local deformations involving at least one unbounded deformation gradient and hence at least one large principal stretch;‡ they include, further, restrictive hypotheses as to the specific response characteristics of the harmonic material in the presence of large extensional deformations.§

The results summarized earlier in this section contain the two as yet undetermined amplitude parameters a_1 and a_2, which cannot be found on the basis of local (asymptotic) considerations. These two parameters are bound to depend on the particular shape of the body and the particular loading to which the latter is subjected; in addition, a_1 and a_2 are in general functionals of the response function H and will involve the material modulus μ. It is apparent from the form of (4.1) that the degenerate case in which $a = 0$ and thus a_1, a_2 vanish simultaneously, is inadmissible: this eventuality is evidently precluded by our assumption regarding the unboundedness of the deformation-gradient tensor as $r \to 0$, upon which the derivation of (4.1) was based.

As far as the main purpose of this study is concerned, the most significant attribute of the asymptotic solution obtained here is that it fails to exhibit the anomalous oscillatory behavior near $r = 0$ displayed by every member of the sequence of solutions to the mixed half-plane problem in the linearized theory, discussed in Section 1.

According to (4.11), the strain-energy density becomes unbounded as $r \to 0$ precisely like r^{-1} for all admissible values of a_1 and a_2. In contrast, the qualitative asymptotic behavior of the spatial coordinates and actual stresses near the origin changes radically depending on whether neither amplitude parameter vanishes or $a_1 = 0$ or $a_2 = 0$. In view of (3.39), the following classification is exhaustive.

$$\text{Case I:} \quad a_1 \ne 0,\ a_2 > 0; \quad \text{Case II:} \quad a_1 = 0,\ a_2 > 0; \quad \text{Case III:} \quad a_1 > 0,\ a_2 = 0. \tag{4.12}$$

Discussion of Case I ($a_1 \ne 0$, $a_2 > 0$)

In dealing with this non-degenerate case we examine first the local character, near $r = 0$, of the deformation image appropriate to the free boundary-leg $\theta = 0$. To this end we write for convenience

$$y_1 = \hat{y}_1(r, 0), \quad y_2 = \hat{y}_2(r, 0) \tag{4.13}$$

and call φ $(0 \le \varphi < 2\pi)$ the angle of inclination at $r = 0$ of the curve into which the ray $\theta = 0$ is carried by the plane deformation under consideration. From (4.2), (4.13) one draws after elementary computations that

$$a_1 = |a| \cos \varphi \ne 0, \quad a_2 = |a| \sin \varphi > 0, \quad |a| = \sqrt{(a_1^2 + a_2^2)} > 0, \tag{4.14}$$

$$y_2 \sim (\tan \varphi) y_1 + \frac{\Delta}{|a|^2} y_1^2, \ \Delta = \Delta(\varphi, \alpha) = \frac{2\alpha - 3 \sin \varphi}{3 \cos^3 \varphi}, \tag{4.15}$$

†Recall that this solution is given by (3.28) with $t = 3/2$.

‡See (3.6) and its implication (3.9), bearing (2.22) in mind.

§See the last of (2.34), which was used in the lowest-order analysis, as well as (3.15) and (3.27), which were employed in the second and third-order analysis, respectively.

y_2 being positive in a neighborhood of $r=0$. Consequently,

$$0<\varphi<\pi/2 \quad \text{if} \quad a_1>0, \quad \pi/2<\varphi<\pi \quad \text{if} \quad a_1<0, \tag{4.16}$$

whereas the curvature at $r=0$ of the deformed free boundary-leg is governed by the sign of Δ. On setting

$$\varphi_o=\sin^{-1}(2\alpha/3), \quad 0<\varphi_o<\pi/2, \tag{4.17}$$

one infers from (4.16) and the second of (4.15) that

$$\left.\begin{array}{c} \Delta=0 \quad \text{if} \quad \varphi=\varphi_o, \quad \varphi=\pi-\varphi_o, \\ \Delta>0 \quad \text{if} \quad 0<\varphi<\varphi_o, \quad \Delta<0 \quad \text{if} \quad \varphi_o<\varphi<\pi/2 \\ \Delta>0 \quad \text{if} \quad \pi/2<\varphi<\pi-\varphi_o, \quad \Delta<0 \quad \text{if} \quad \pi-\varphi_o<\varphi<\pi. \end{array}\right\} \tag{4.18}$$

The geometric alternatives corresponding to (4.16), (4.18) are illustrated in Fig. 2.

According to (4.7) all three actual stress components in Case I become unbounded as $r\to 0$ precisely like $r^{-1/2}$, both normal stresses being tensile at all material points sufficiently close to the origin. Moreover, if τ_1, τ_2 ($\tau_1>\tau_2$) are the principal Cauchy stresses and ϕ_1, ϕ_2 the inclinations of the associated principal axes, one finds

$$\tau_1(r,\theta)\sim\frac{\mu|a|^2}{2p(\theta)}r^{-1/2}, \quad \tau_2(r,\theta)=o(r^{-1/2}), \quad \phi_1(r,\theta)=\phi_2(r,\theta)-\frac{\pi}{2}\sim\varphi, \tag{4.19}$$

so that in particular

$$\tau_1(r,0)\sim\frac{\mu|a|}{\alpha}r^{-1/2}, \quad \phi_1(r,0)\sim\varphi. \tag{4.20}$$

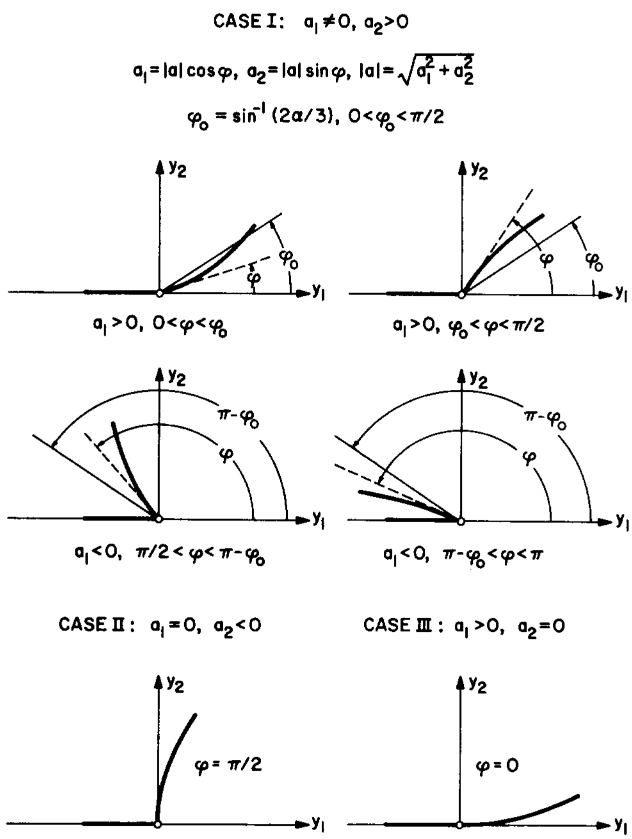

Fig. 2. Deformation image of boundary near origin.

The second of (4.20) reflects the fact that the deformation image of the ray $\theta = 0$ is traction-free and hence must be a trajectory of the actual principal stresses.

Asymptotic estimates for the principal stretches λ_1, λ_2 ($\lambda_1 > \lambda_2$) are easily deduced in Case I from (3.13) by recourse to (3.2) and (2.22). In this manner one arrives at

$$\lambda_1(r,\theta) \sim \frac{|a|}{2} r^{-1/2}, \quad \lambda_2(r,\theta) \sim \frac{2p(\theta)}{|a|}. \tag{4.21}$$

From (4.20), (4.21) follows

$$\tau_1(r,0) \sim \frac{2\mu}{\alpha} \lambda_1(r,0), \quad \lambda_2(r,0) \sim \alpha, \tag{4.22}$$

as was to be anticipated on the basis of (2.51) since $\lambda_1(r, 0)$ becomes unbounded as $r \to 0$ and the state of stress at all material points corresponding to $\theta = 0$ is one of uni-axial tension parallel to the deformed free leg of the boundary.

Discussion of Case II ($a_1 = 0$, $a_2 > 0$)

In discussing the two degenerate cases we continue to employ the simplified notation introduced by (4.13); further, φ is understood to be the angle defined in the discussion of Case I.

Equations (4.3), which pertain to Case II, yield

$$y_2 \sim \frac{a_2}{(1-2\alpha/3)^{1/2}} y_1^{1/2}, \ \varphi = \pi/2, \tag{4.23}$$

whence the deformation image of the boundary-ray $\theta = 0$ near $r = 0$ is in first approximation a parabola with a vertical tangent at the origin (see Fig. 2). Since $0 < \alpha < 1$, both y_1 and y_2 are positive close enough to the origin.

Equations (4.9) supply the dominant asymptotic behavior at the fixed boundary-leg $\theta = \pi$ of the actual stresses in Case II. Evidently the most important of these stresses is the tensile stress $\tau_{22}(r, \pi)$, which becomes infinite like $r^{-1/2}$ at the origin; the normal stress $\tau_{11}(r, \pi)$ is also tensile, but tends to zero as $r \to 0$, while the shear stress $\tau_{12}(r, \pi)$ remains finite in this limit.

Discussion of Case III ($a_1 > 0$, $a_2 = 0$)

In this instance one has according to (4.4),

$$y_2 \sim \frac{2\alpha}{3a_1^2} y_1^2, \quad \varphi = 0, \tag{4.24}$$

both y_1 and y_2 being positive near the origin. Thus the leg $\theta = 0$ is locally carried into a parabolic arc with a horizontal tangent at $r = 0$ (see Fig. 2).

One infers from (4.10) that in Case III the predominant stress at $\theta = \pi$ is $\tau_{11}(r, \pi)$, which is tensile and unbounded like r^{-1} as $r \to 0$; $\tau_{22}(r, \pi)$ is tensile and finite, whereas the shear stress $\tau_{12}(r, \pi)$ becomes infinite as $r \to 0$ merely like $r^{-1/2}$.

As has been pointed out already, the values of the amplitude parameters a_1 and a_2, which govern the scale of the near-field solution under discussion, cannot be extracted from a purely local asymptotic analysis. We turn now to a very special problem of some practical interest, which involves a fixed-free rectilinear boundary-segment and for which $|a|$ may be determined directly from the data by means of the conservation law (2.14) under certain physically plausible assumptions regarding the behavior of the unknown solution at infinity, as well as in the vicinity of the singular boundary point.†

Thus consider a doubly infinite strip of width l (see Fig. 3), the upper edge of which is free of loading, while two adjoining semi-infinite segments of the lower edge are free and fixed, respectively. The semi-infinite portion of the strip that possesses load-free edges is subjected to

†See Rice[14] for related examples illustrating the direct use of the conservation law.

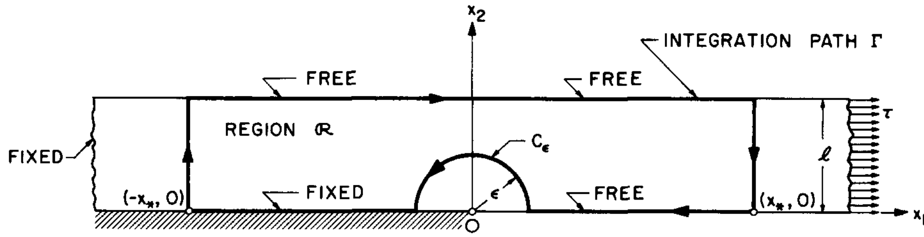

Fig. 3. Strip problem, geometry and coordinates.

uniform uni-axial tractions (parallel to the edges) at infinity; the other half of the strip is immobilized at infinity. For the choice of the coordinate frame shown in Fig. 3 the material points of the strip (or rather of its middle section) in the undeformed configuration occupy the closure of the region $\mathcal{R}$ defined by

$$\mathcal{R} = \{(x_1, x_2) | -\infty < x_1 < \infty, \quad 0 \le x_2 \le l, \quad x_1^2 + x_2^2 \neq 0\}, \tag{4.25}$$

from which the origin has been omitted. The boundary conditions described above may be written as†

$$\left.\begin{aligned} \sigma_{\alpha 2}(x_1, l) = 0 \quad (-\infty < x_1 < \infty), \\ u_\alpha(x_1, 0) = 0 \quad (-\infty < x_1 < 0), \quad \sigma_{\alpha 2}(x_1, 0) = 0 \quad (0 < x_1 < \infty). \end{aligned}\right\} \tag{4.26}$$

On the other hand the prescribed behavior at infinity demands

$$\left.\begin{aligned} u_\alpha(x_1, x_2) = o(1) \quad \text{as} \quad x_1 \to -\infty \quad (0 \le x_2 \le l), \\ \tau_{\alpha\beta}(x_1, x_2) = \tau\delta_{\alpha 1}\delta_{\beta 1} + o(1) \quad \text{as} \quad x_1 \to +\infty \quad (0 \le x_2 \le l), \end{aligned}\right\} \tag{4.27}$$

where $\tau > 0$ is the intensity of the applied loading, and (4.27) will be taken to hold uniformly with respect to x_2. The over-all equilibrium of the strip necessitates that $\sigma_{12}(x_1, 0)$ and $x_1\sigma_{22}(x_1, 0)$ be integrable for $-\infty < x_1 \le 0$. We shall further restrict the admissible field behavior near $r = 0$ by requiring that

$$u_\alpha(x_1, x_2) = O(1), \quad \sigma_{\alpha\beta}(x_1, x_2) = O(r^{-\delta}) \quad \text{as} \quad r \to 0 \quad (\delta < 1). \tag{4.28}$$

In what follows we presuppose the existence of a solution (suitably smooth on $\mathcal{R}$) to the equations of finite plane strain for a harmonic material on interior of $\mathcal{R}$ that meets conditions (4.26) to (4.28). Moreover, it is natural to anticipate that the deformation field of such a solution uniformly tends to the undeformed state as $x_1 \to -\infty$ and to a homogeneous state of uni-axial tension of intensity τ, parallel to the x_1-axis, as $x_1 \to \infty$. Bearing (2.1), (2.2) and (2.31), (2.44) in mind, we thus have

$$\left.\begin{aligned} u_{\alpha,\beta}(x_1, x_2) = o(1) \quad \text{as} \quad x_1 \to -\infty \quad (0 \le x_2 \le l), \\ [u_{\alpha,\beta}(x_1, x_2)] = \begin{bmatrix} \lambda - 1 & 0 \\ 0 & \bar{\lambda}(\lambda) - 1 \end{bmatrix} + o(1) \quad \text{as} \quad x_1 \to \infty \quad (0 \le x_2 \le l), \end{aligned}\right\} \tag{4.29}$$

where the axial stretch λ and the transverse stretch $\bar{\lambda}(\lambda)$ are linked to the loading τ through

$$\tau = 2\mu\left[\frac{\lambda}{\bar{\lambda}(\lambda)} - 1\right], \quad H'(\lambda + \bar{\lambda}(\lambda)) = \lambda \tag{4.30}$$

and the material is of course assumed to admit a regular state of uni-axial tension in plane strain.

At this stage we apply the conservation law (2.14), with $\alpha = 1$ and for the path of integration

†Recall from (2.13) that the Piola traction-vector must vanish at a free boundary.

$\Gamma = \Gamma(\epsilon, x_*)$ displayed in Fig. 3, to the solution field of the problem under consideration. Accordingly,

$$\int_\Gamma h(x_1, x_2)\, d\mathcal{S} = 0, \quad h = Wn_1 - \sigma_{\beta\rho} n_\rho u_{\beta,1} \quad \text{on} \quad \Gamma, \tag{4.31}$$

where $\mathbf{n}$ is the outward unit normal vector of the closed contour Γ. Along the two rectilinear segments of Γ that coincide with the edges of the strip, $n_1 = 0$ and either $\sigma_{\beta 2}$ or $u_{\beta,1}$ vanishes because of the boundary conditions (4.26). Therefore (4.31) yields

$$\int_{C_\epsilon} h(x_1, x_2)\, d\mathcal{S} = -\int_0^l h(-x_*, x_2)\, dx_2 - \int_0^l h(x_*, x_2)\, dx_2 \quad (0 < \epsilon < l) \tag{4.32}$$

for every $x_* > 0$, where C_ϵ is the semi-circular portion of Γ traversed in a clockwise sense (see Fig. 3).

Proceeding to the limit as $x_* \to \infty$ one finds that the first integral in the right member of (4.32) tends to zero on account of the first of (4.29), since W vanishes in the undeformed state. The corresponding limit of the second integral may be evaluated with the aid of (4.27), (4.29), (4.30), (4.31) and (2.6), (2.15), (2.22). In this manner one is led to

$$\begin{aligned} \mathcal{J}(\tau) &= \lim_{\epsilon \to 0} \int_{C_\epsilon} h(x_1, x_2)\, d\mathcal{S} = -2\mu l[H(\lambda + \bar{\lambda}(\lambda)) - \bar{\lambda}(\lambda) - \lambda(\lambda - 1)] \\ &= 2\mu l \int_1^{\lambda(\tau)} (\xi - 1)[1 - \bar{\lambda}'(\xi)]\, d\xi \quad (0 < \tau < \infty); \end{aligned} \tag{4.33}$$

the dependence of λ upon the load intensity τ is implicit in (4.30),† the second of which justifies the integral representation for $\mathcal{J}(\tau)$ in (4.33). This integral representation, in turn, makes it obvious that‡ $\mathcal{J}(\tau) > 0$.

Suppose now, as is reasonable, that the solution to the strip problem near the origin admits the asymptotic representation (3.6). Then the limit occurring in (4.33) is computable on the basis of the lowest-order asymptotic solution (3.13) by means of the definition of h in (4.31) and by appealing to (4.11), (2.62) and (2.63). This elementary calculation gives

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} h(x_1, x_2)\, d\mathcal{S} = \frac{\mu\pi}{4}(a_1^2 + a_2^2), \tag{4.34}$$

and combining (4.34) with (4.33) one has

$$|a|^2 = a_1^2 + a_2^2 = \frac{4}{\mu\pi}\mathcal{J}(\tau) = -\frac{8l}{\pi}[H(\lambda + \bar{\lambda}(\lambda)) - \bar{\lambda}(\lambda) - \lambda(\lambda - 1)] \quad (0 < \tau < \infty), \tag{4.35}$$

where $\lambda = \lambda(\tau)$ and $\bar{\lambda}(\lambda)$ are once again determined by (4.30). Equations (4.35), (4.30), together with (2.25), permit one to deduce the *small-load estimate*

$$a_1^2 + a_2^2 = \frac{(1 - \nu)l}{\pi\mu^2}\tau^2 + o(\tau^2) \quad \text{as} \quad \tau \to 0, \tag{4.36}$$

where μ and ν are the shear modulus and Poisson's ratio of the material for infinitesimal deformations. In contrast, if the harmonic material is such that its response to large homogeneous uni-axial tension (under plane strain) obeys (2.51), then (4.35), (4.30), (2.46) lead to the *large-load estimate*

$$a_1^2 + a_2^2 = \frac{\alpha^2 l}{\pi\mu^2}\tau^2 + o(\tau^2) \quad \text{as} \quad \tau \to \infty; \tag{4.37}$$

the physical significance of the material parameters μ and α in the present context is clear from (2.51).

†Recall from the discussion of uni-axial tension in Section 2 that $\tau = \bar{\tau}(\lambda)$ is monotone increasing and hence invertible.

‡Recall from Section 2 that $\bar{\lambda}(\lambda)$ is steadily decreasing and that $\lambda(\tau) > 1$ or $\lambda(\tau) < 1$ according as $\tau > 0$ or $\tau < 0$. Thus $\mathcal{J}(\tau)$ is positive also in the presence of a compressive loading at infinity, i.e. for $\tau < 0$.

Note that (4.35), as well as the estimates (4.36), (4.37), supply merely the sum of the squares of the two amplitude parameters but fail to furnish a_1 and a_2 individually. Thus, while the conservation law (2.14) renders the estimate (4.11) for the strain-energy density W fully determinate, it does not lead to a complete determination of the dominant asymptotic behavior appropriate to the deformation and stresses in general.

We proceed now to the counterpart of the foregoing strip problem in the linearized theory of elastostatic plane strain. If $\mathring{u}_\alpha$ and $\mathring{\sigma}_{\alpha\beta}$ stand for the cartesian components of displacement and stress in the latter theory, the linear analogue of the problem at hand may be stated as follows. One is to find fields $\mathring{u}_\alpha$, $\mathring{\sigma}_{\alpha\beta}$, suitably smooth on $\mathscr{R}$, that satisfy the linear plane-strain equations on the interior of $\mathscr{R}$ and conform to (4.26), (4.27), (4.28) with $\mathring{u}_\alpha$ and $\mathring{\sigma}_{\alpha\beta}$ in place of u_α and $\sigma_{\alpha\beta}$; in addition, $\tau_{\alpha\beta}$ in the last of (4.27) is now to be replaced† by $\mathring{\sigma}_{\alpha\beta}$.

This boundary-value problem is one of considerable analytical complexity, which—so far as we are aware—has not been solved. Our present interest in this connection is confined to deducing the analogue of (4.33) in the linear theory for the purpose of examining its relation to (4.33) in the limit as the assigned load intensity τ tends to zero.

With such an objective in mind we observe first that (4.29) in the current circumstances give way to

$$\left.\begin{aligned} \mathring{u}_{\alpha,\beta}(x_1,x_2) &= o(1) \quad \text{as} \quad x_1\to-\infty \quad (0\le x_2\le l),\\ [\mathring{u}_{\alpha,\beta}(x_1,x_2)] &= \begin{bmatrix} (1-\nu)\tau/2\mu & 0\\ 0 & -\nu\tau/2\mu \end{bmatrix} + o(1) \quad \text{as} \quad x_1\to\infty \quad (0\le x_2\le l), \end{aligned}\right\} \tag{4.38}$$

provided μ and ν are the shear modulus and Poisson's ratio of the linearly elastic material, and (4.38) are taken to hold uniformly with respect to x_2. Moreover, (4.38) assure that

$$\left.\begin{aligned} \mathring{W}(x_1,x_2) &= o(1) \quad \text{as} \quad x_1\to-\infty \quad (0\le x_2\le l),\\ \mathring{W}(x_1,x_2) &= \frac{(1-\nu)\tau^2}{4\mu} \quad \text{as} \quad x_1\to\infty \quad (0\le x_2\le l), \end{aligned}\right\} \tag{4.39}$$

uniformly in x_2, if $\mathring{W}$ denotes the strain-energy density based on the linear theory of plane strain. Finally, from the counterpart of the conservation law (2.14) in the linear theory follows, upon setting $\alpha = 1$,

$$\int_\Gamma \mathring{h}(x_1,x_2)\,d\mathscr{S} = 0, \quad \mathring{h} = \mathring{W}n_1 - \mathring{\sigma}_{\beta\rho}n_\rho\mathring{u}_{\beta,1} \quad \text{on} \quad \Gamma, \tag{4.40}$$

where Γ is any closed regular path in $\mathscr{R}$ and n_ρ are the components of its outward unit normal vector.

Choosing for Γ the closed contour indicated in Fig. 3 and proceeding precisely as before, one obtains upon going to the limit first as $x_*\to\infty$ and then as $\epsilon\to 0$,

$$\mathring{\mathscr{J}}(\tau) = \lim_{\epsilon\to 0}\int_{C_\epsilon} \mathring{h}(x_1,x_2)\,d\mathscr{S} = \frac{l(1-\nu)\tau^2}{4\mu}. \tag{4.41}$$

On the other hand, expanding $\mathscr{J}(\tau)$ given by (4.33) in a Taylor series about $\lambda = 1$ ($\tau = 0$) and appealing once more to (4.30) and (2.25), one finds that

$$\mathscr{J}(\tau) = \mathring{\mathscr{J}}(\tau) + o(\tau^2) \quad \text{as} \quad \tau\to 0. \tag{4.42}$$

It should be emphasized that this conclusion was reached without any assumptions concerning the approximative status of the solution to the linearized strip problem in its relation to the solution of the corresponding nonlinear problem.

Let us next return briefly to the problem of the bonded punch (see Fig. 1), the formulation and solution of which within the linear theory of elastostatic plane strain was discussed in Section 1. Suppose $P < 0$, so that the given scalar punch load is tensile, rather than compressive. Then the

†Recall that the actual and the Piola stresses merge in the transition to the linear theory. See the remarks following (2.14).

local behavior of the solution to the corresponding problem in the nonlinear theory of harmonic materials should be furnished by the asymptotic results assembled at the beginning of this section, provided a_1 and a_2 are suitably determined. The requisite determination of the two amplitude parameters again eludes a purely local asymptotic analysis. Nor is the procedure used in the strip problem to find $|a|$ for *all* values of the prescribed loading from the known far-field behavior applicable in the present circumstances. It is possible, however, to deduce a *small*-load estimate for $|a|$ from the available solution to the linearized punch problem with the aid of an assumption suggested by (4.42).

Let C_ϵ at present denote the semi-circular arc of radius ϵ centered at the right-hand punch corner (see Fig. 1) and let

$$\mathcal{J}(P)=\lim_{\epsilon\to 0}\int_{C_\epsilon} h(x_1,x_2)\,\mathrm{d}\mathcal{S},\quad \mathring{\mathcal{J}}(P)=\lim_{\epsilon\to 0}\int_{C_\epsilon} \mathring{h}(x_1,x_2)\,\mathrm{d}\mathcal{S}, \tag{4.43}$$

where h and $\mathring{h}$ are defined as in (4.31) and (4.40) in terms of the solution fields corresponding to the finite and the infinitesimal theory, n_ρ being the components of the unit normal vector of C_ϵ that points toward the center of the arc. The analogue of (4.42) in the punch problem is

$$\mathcal{J}(P)=\mathring{\mathcal{J}}(P)+o(P^2)\quad \text{as}\quad P\to 0. \tag{4.44}$$

Suppose (4.44) holds true. Now, $\mathcal{J}(P)$ is computable on the basis of our lowest-order asymptotic solution to the nonlinear problem, which yields (4.34). On the other hand, $\mathring{\mathcal{J}}(P)$ may be calculated from Muskhelishvili's [7] solution of the linearized punch problem.† In this manner one arrives at

$$\mathcal{J}(P)=\frac{\mu\pi}{4}(a_1{}^2+a_2{}^2),\quad \mathring{\mathcal{J}}(P)=\frac{(1-\nu)P^2}{4\pi\mu l} \tag{4.45}$$

and thus (4.44) leads to the estimate

$$a_1{}^2+a_2{}^2+\frac{1-\nu}{\pi^2\mu^2 l}P^2+o(P^2)\quad \text{as}\quad P\to 0. \tag{4.46}$$

This result, like its counterpart (4.36) in the strip problem, fails to characterize a_1 and a_2 individually.

The foregoing derivation of (4.46) is contingent upon the validity of (4.44). In contrast to (4.42), the relation (4.44) cannot be confirmed directly from the data of the nonlinear and the linear problem under present consideration. We show now that (4.44) may, however, be deduced with the aid of the conservation law (2.14) and the analogous law in the infinitesimal theory on the basis of a physically convincing assumption regarding the approximative role of the solution to the linearized bonded-punch problem.

For this purpose let Γ currently stand for the closed contour shown in Fig. 1 and let $\mathbf{n}$ be the unit outward normal vector of Γ. Then (2.14) and the corresponding conservation law for linearized plane strain assure that (4.31) and (4.40) hold also in the present instance provided the ingredients of h and $\mathring{h}$ refer to the appropriate solution field of the punch problem. Further, in view of the prevailing boundary conditions,‡ the horizontal portion of Γ does not contribute to the integrals in (4.31) and (4.40). Suppose at this stage that the solution of the problem in the linear theory approximates its counterpart in the finite theory to the extent that

$$u_{\alpha,\beta}(x_1,x_2)=\mathring{u}_{\alpha,\beta}(x_1,x_2)+o(P^2)\quad \text{as}\quad P\to 0 \tag{4.47}$$

uniformly on the vertical component of Γ and on the quarter-circle C_ρ (see Fig. 1) for a sufficiently large radius ρ. It then follows readily from (4.31), (4.40) that

$$\int_{C_\epsilon} h(x_1,x_2)\,\mathrm{d}\mathcal{S}=\int_{C_\epsilon}\mathring{h}(x_1,x_2)\,\mathrm{d}\mathcal{S}+o(P^2)\quad \text{as}\quad P\to 0 \tag{4.48}$$

for every small enough $\epsilon>0$, and (4.48), (4.43) yield (4.44) upon passage to the limit as $\epsilon\to 0$.

†We omit the details of this quite cumbersome calculation.
‡Recall (1.21), which apply equally to the nonlinear punch problem if $\sigma_{\alpha 2}$ there are interpreted as Piola stress components.

It should be emphasized that (4.47) cannot possibly be valid at *all* material points (x_1, x_2) other than the punch corners since the solution based on the linearized theory, as pointed out in Section 1, is oscillatory in r near† $r = 0$, whereas the asymptotic field behavior, as $r \to 0$, furnished by the finite theory of harmonic materials, which was established in Section 3, is free from such oscillations. For the same reason (4.47) cannot hold uniformly on *every* closed set of material points that has a *finite distance* from both punch corners.

The preceding observations do not, however, destroy the plausibility of the assumption introduced in connection with (4.47), which refers merely to material points *sufficiently distant* from the corners of the punch. Indeed, if this assumption were to be false, it would be difficult to attach any practical importance to the distribution of contact stresses predicted by the linear theory of elasticity.

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†Here r is once again the distance from the right-hand corner of the punch (see Fig. 1).